Wednesday, December 30, 2015 4:31 PM

every $g \neq 1$ has no fixed pt.

- **Main theorem**: G acts **freely and properly** => M/G is smooth manifold and quotient map is submersion.
- G acts continuously on M => **M->M/G is an open map** $\pi^{-1}(\pi(\underbrace{U}_{\text{open}})) = \bigcup_{g \in G} g \cdot U$ is open.
- **Proper action**: (g.p,p): G * M -> M * M is a proper map (weaker than G * M -> M being proper) $G \times M \to M \times M \xrightarrow{pr_1} M$
- Eg. $\mathbb{R} \curvearrowright T^2$ irrationally is not proper
- **Proper action => M/G is Hausdorff** $\leftarrow$ proper $\Rightarrow$ closed map $\Rightarrow \{(g.p,p)\} \subset_{\text{closed}} M \times M \Rightarrow$ orbit$_p$ is closed $\forall p$

  $\Rightarrow$ orbit$_p$ & orbit$_q$ sep$^{\text{td}}$ by open sets

  $\Rightarrow$ $[p]$ & $[q]$ sep$^{\text{td}}$ by open sets since $M \to M/G$ is open.
- Proper action

  $\Leftrightarrow (p_i)$ & $(g_i p_i)$ converge $\Rightarrow$ subseq of (g_i) conv.

  $\Leftrightarrow \forall$ cpt K, $\{g \in G : (g \cdot K) \cap K \neq \phi\}$ is cpt.

  (called properly discont. if G is discrete)

To check $f(S)$ is closed. $f^{-1}(\bar{u})$ cpt $\Rightarrow f^{-1}(\bar{u}) \cap \underset{\text{closed}}{S}$ cpt.

$\underset{\text{open w/ } \bar{U} \text{ cpt}}{U} \ni p \in \overline{f(S)}$. $\Rightarrow p \in \overline{U \cap f(S)}$ $\Rightarrow \bar{U} \cap f(S)$ cpt. hence closed $\Rightarrow p \in f(S)$.

- **Action of compact Lie group is proper**
- SL(2,Z) action on R^2 is not proper! (K={0})
- **Proper => orbit is closed embedded submanifold**
- **Isotropy group at p is compact**

$\{g : g \cdot p = p\} = \{g \cdot \{p\} \cap \{p\} \neq \phi\}$

Trivial action has closed orbits but not proper!

$G \xrightarrow{\theta} M$ is proper (and hence closed):
$g \longmapsto g \cdot p$

$\underset{\text{cpt.}}{K} \subseteq M$. $\theta^{-1}(K) = \{(g \cdot \{p\}) \cap K \neq \phi\}$ is cpt.

$\bar{\theta} : G/G_p \xrightarrow{G\text{ equiv.}} M$ is inj. imm.

mfd by main thm. $G \curvearrowleft G_p$ cpt.

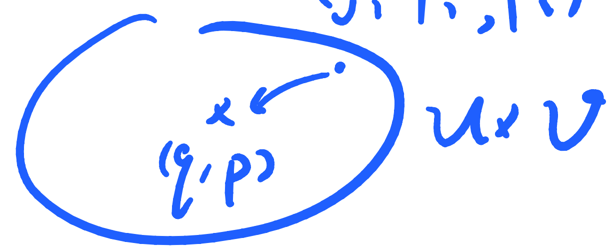
$\bar{\theta}$ is $\begin{cases} \text{inj.} \\ \text{imm.} \\ \text{and closed} \end{cases} \Rightarrow$ embedded. $(\text{Dom} \xrightarrow[\text{homeo.}]{} \text{Im})$

$G\times M \xrightarrow[\pi]{} M\times M$ proper $\Rightarrow$ $\left[(p_i)\ \&\ (g_i p_i) \text{ converge} \Rightarrow \text{subseq of } (g_i) \text{ conv.}\right]$:

$p = \lim p_i$, $q = \lim g_i \cdot p_i$. $(q,p) \in \mathcal{U}\times\mathcal{V} \subset M\times M$ s.t. $\overline{\mathcal{U}\times\mathcal{V}}$ cpt.

$(g_i, p_i) \in \pi^{-1}(\overline{\mathcal{U}\times\mathcal{V}})$ cpt $\Rightarrow$ subseq. conv.

$$\pi(g_i, p_i) = (g_i\cdot p_i, p_i)$$



$\left[(p_i)\ \&\ (g_i p_i) \text{ converge} \Rightarrow \text{subseq of } (g_i) \text{ conv.}\right]$ $\Rightarrow$ $\forall$ cpt K, $\{g\in G : (g\cdot K_1)\cap K_2 \neq \phi\}$ is cpt. :
(or : K_1, K_2)

$\vdash$: any seq. g_i : $(g_i \cdot K_1)\cap K_2 \neq \phi$ has conv. subseq.

$$K_2 \ni q_i = g_i\cdot p_i, \quad p_i \in K_1$$

p_i, $g_i\cdot p_i$ conv. in subseq. $\Rightarrow$ g_i conv. in subseq.

$\forall$ cpt K, $\{g\in G : (g\cdot K_1)\cap K_2 \neq \phi\}$ is cpt. $\Rightarrow$ $G\times M \xrightarrow[\pi]{} M\times M$ proper :

$$M\times M \supset_{\text{cpt}} C$$

(or $K_i = \pi_i(C)$)

$K \triangleq \pi_1(C)\cup\pi_2(C)$ cpt . $\pi^{-1}(K_1\times K_2) = \{(g,p) : (g\cdot p, p)\in K_1\times K_2\} \underset{\text{closed}}{\subset} \underbrace{\{g : (g\cdot K_2)\cap K_1 \neq \phi\}\times K_2}_{\text{cpt}}$

$\therefore$ $\pi^{-1}(K_1\times K_2)$ cpt.

$\cup$ closed

$\pi^{-1}(C)$ $\Rightarrow$ cpt.

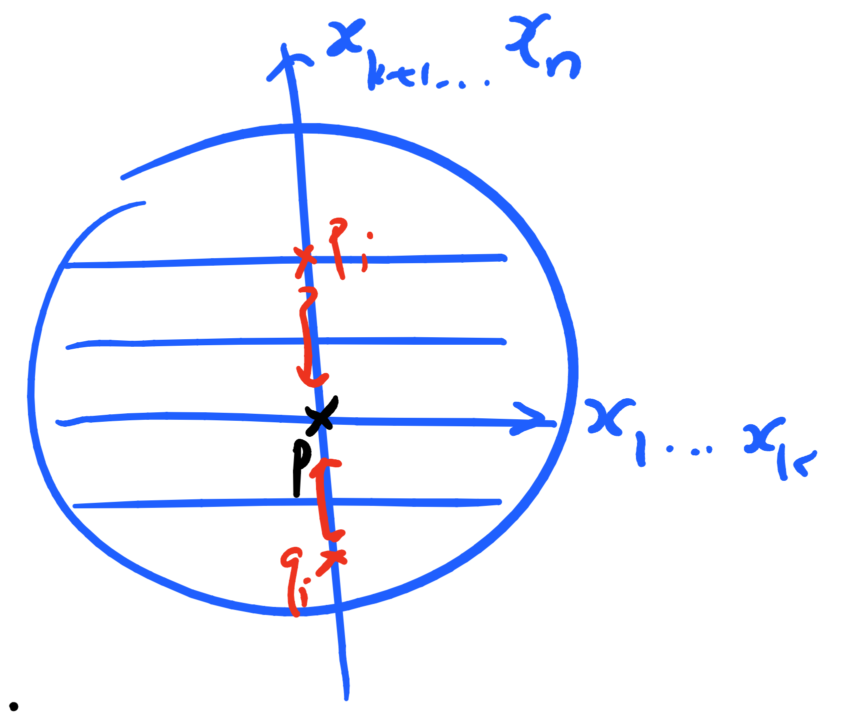
Pf of main thm.:

free action $\Rightarrow$ orbit distr. has rank $k = \dim G$. (If $X_\#(p)=0$, p is fixed by $e^{tX_\#}$.)

$\mathfrak{g} \curvearrowright M \xrightarrow[\substack{\text{Lie homo.}\\ \text{const rk}}]{}$ int. distribution. Leaves are orbits.

Loc. coord. of $M/G \ni [p]$: take coord. $(x_1 \dots x_n)$ around p adapted to the orbit foliation:

orbits $= \{x_{k+1}, \dots, x_n$ are const.$\}$



Want: $(x_{k+1} \dots x_n) : M/G \xrightarrow{\text{cont.}} \mathbb{R}^{n-k}$ gives loc. coord

injective: by shrinking the chart if necessary, different $(x_{k+1}, \dots, x_n)$ corr. to different orbits.

If not, $\exists\, p_i, q_i \to p$ (in $\{x_1 = \dots = x_k = 0\}$) in same orbit but $p_i \neq q_i$.

$q_i = g_i p_i$

$\Rightarrow g_i \to g$ by properness.

$g \cdot p = p \Rightarrow g = 1$ by freeness.

But $G \times \{x_1 = \dots = x_k = 0\} \longrightarrow M$, $(g, q) \mapsto g \cdot q$ is loc. diffeo. $\therefore g_i p_i = 1 \cdot q_i \Rightarrow g_i = 1$. $p_i = q_i$

$\rightarrow \leftarrow$

Suppose $(x_1 \dots x_n)$ & $(y_1 \dots y_n)$ are adapted coord.

Then y_{k+i} are indep. of $(x_1 \dots x_k)$: varying $(x_1 \dots x_k)$ stays in same orbit and hence y_{k+i} remains the same.

Thus have change of coord. $\left(y_i(x_{k+1} \dots x_n)\right)_{i=k+1}^{n}$ for M/G.

$M \xrightarrow[(x_{k+1} \dots x_n)]{} M/G$ is a submession. #

Examples.

- $\mathbb{R}^2/S^1 \simeq \mathbb{R}_{\geq 0}$.
- $\mathbb{R}^n / GL(n, \mathbb{R}) = \{[0], [v \neq 0]\}$. Not Hausdorff.
- $S^3/S^1 = S^2$. $\lambda \cdot (z, w) = (\lambda z, \lambda w)$

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \text{Hopf fibration} \\ & & S^2 \end{array}$$

- $\underset{G=}{\mathbb{R}} \overset{\text{transl.}}{\curvearrowright} \underset{M=}{\mathbb{R}^2}$ is proper. But $G \times M \to M$ is not: orbits are non-cpt.