Friday, April 1, 2016 9:09 PM

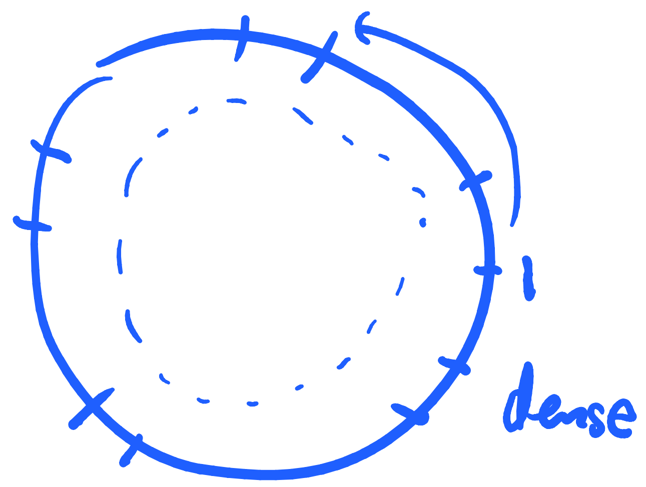
Prop.: G discrete $\overset{\text{proper}}{\curvearrowright} M$ (also called properly '$\underbrace{\text{discontinuous}}_{\text{discrete}}$')

$$\iff \begin{cases} 1.\ \forall p,\ \exists \overset{\text{open}}{U} \ni p \text{ st } g\cdot U \cap U = \emptyset \quad \forall g \neq 1. \\ 2.\ \text{If } p, p' \text{ are not in same orbit, then} \\ \quad \exists \underset{\text{open}}{V} \ni p,\ \underset{\text{open}}{V'} \ni p' \text{ st. } (g\cdot V)\cap V' = \emptyset \quad \forall g. \end{cases}$$

(quotient is Hausdorff)

e.g. discrete free but quotient is not a manifold:

$\mathbb{Z} \curvearrowright S^1$, $\quad k\cdot z = e^{2\pi i\theta\cdot k}\cdot z$ where $\theta \notin \mathbb{Q}$.



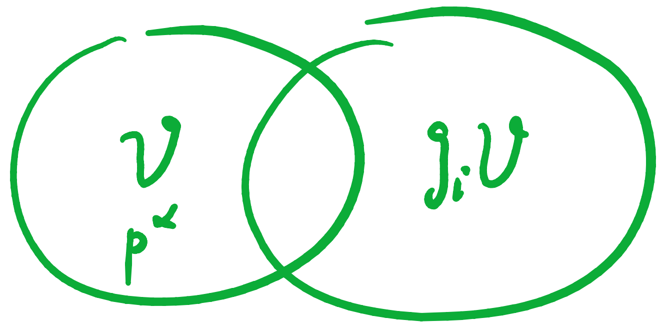
Pf. $\Rightarrow$) 2. Hausdorff ✓.

1. Take $V \ni p$ w/ $\bar{V}$ cpt. $\exists$ only finitely many $g_1 \cdots g_k \neq 1$ w/ $(g_i \cdot \bar{V}) \cdot \bar{V} \neq \phi$.

Make $g_i \cdot \bar{V} \not\ni p$: shrink V s.t. $\bar{V} \not\ni g_i^{-1} \cdot p$ $\forall i$.

Then take $U = V - \bigcup_i g_i \cdot \bar{V}$.

$$U \cap g \cdot U \subset U \cap g \cdot V = \phi.$$



$\Leftarrow$) Consider $p_i \to p$. $g_i \cdot p_i \to q$. Want g_i conv. in subseq.

If p & q not in the same orbit,

$\exists$ $\underset{\substack{\cup \\ p_i}}{V_p}$, $\underset{\substack{\cup \\ g_i p_i}}{V_q}$ s.t. $g \cdot V_p \cap V_q = \phi$ $\forall g$.

$g_i \cdot p_i \in g_i V_p \cap V_q$ $\forall i \gg 0$. $\rightarrow\leftarrow$

$\therefore$ $q = g \cdot p \leftarrow g_i \cdot p_i$. Want $g_i \to g$ in subseq.

$(g^{-1} g_i) p_i \to p$.

$\exists$ U_p s.t. $h \cdot U_p \cap U_p = \phi$ $\forall h \neq 1$.

$(g^{-1} g_i) \cdot U_p \cap U_p \ni (g^{-1} g_i) \cdot p_i$ $\forall i \gg 0$.

$\therefore$ $g^{-1} g_i = 1$, i.e. $g_i = g$ $\forall i \gg 0$.

Prop. $E \xrightarrow{\pi} M$ covering map.

Then $\underbrace{\mathrm{Aut}(E \longrightarrow M)}_{\text{discrete}}$ acts freely & properly on E.

(don't know transitive or not)

Pf: free:

$\varphi \in \mathrm{Aut}(E \xrightarrow{\pi} M)$:

$$\begin{tikzcd} & E \arrow[d] \\ E \arrow[ur, "\varphi"] \arrow[r] \arrow[ur, phantom, "\circlearrowright", shift right=3] & M \end{tikzcd}$$

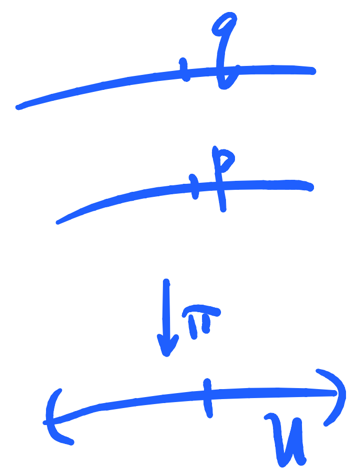
regarded as lift.

If $\varphi(p) = p = \mathrm{Id}(p)$, then $\varphi = \mathrm{Id}$ by uniqueness of lift.

Hausdorff:

If p, q not in same orbit, obvious if $\pi(p) \neq \pi(q)$.

If $\pi(p) = \pi(q)$, take $U \ni \pi(p)$ s.t. $\pi^{-1}(U) = \coprod_{p' \in \pi^{-1}\{\pi(p)\}} U_{p'}$.



p, q belong to different orbits

$$\Rightarrow U_{g \cdot p} \cap U_q = \phi \quad \forall g.$$

$\underline{g \cdot U_p \cap U_p = \phi}$ $\forall g \neq 1$ since g acts freely.

Prop. $\underset{\text{discrete}}{G} \xrightarrow{\text{free proper}} M$. Then $M \longrightarrow M/G$ is a covering.

Pf: known: submersion.

For $[p] \in M/G$, take $U_p \subset M$ s.t. $U_p \cap g \cdot U_p = \phi \quad \forall g \neq 1$.

Then $\pi^{-1}(\pi(U_p)) = \coprod_g \underbrace{g \cdot U_p}_{\simeq U_p}$ and $U_p \simeq \pi(U_p)$.