- Def: M admits a transitive action by a Lie group
- Examples: sphere = O(n)/O(n-1), upper half plane = SL(2,R)/SO(2), CP^n = U(n+1)/U(1)U(n), Gr(k,n) = U(n)/U(k)U(n-k) $= GL(n)/\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$.
  Flag manifold = GL(n)/$\left(\begin{smallmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{smallmatrix}\right) = U(n)/U(k_1)\cdots U(k_l)$ $(k_1+\cdots+k_l = n.)$ Kähler
- G acts transitively on M <-> M = G / H for some closed subgroup H
- G acts transitively on set with closed isotropy subgp => S is a manifold.

- $G/\underbrace{H}_{\text{closed}}$ is a homogeneous sp.

Pf.: Obviously it has a left transitive G-action.

It is a mfd: $G \curvearrowleft H$ freely: $g \cdot h = g \Rightarrow h = 1$.

proper: $g_i \to g$, $g_i h_i \to \tilde{g}$ $\Rightarrow h_i = g_i^{-1}(g_i h_i) \to g^{-1}\tilde{g} \in H$. (H is closed)

G-action is sm.:

$$\begin{tikzcd} G \times G \arrow[r, "\varphi"] \arrow[d, "\pi_1"] \arrow[dr, "\pi_2 \circ \varphi"] & G \arrow[d, "\pi_2"] \\ G \times (G/H) \arrow[r, "\bar{\varphi}"] & G/H \end{tikzcd}$$

$\pi_2 \circ \varphi$ sm. $\Longrightarrow \bar{\varphi}$ sm. since π_1 is a submer.
(loc. coord comes from upstairs) #

- A homogeneous G-mfd M is G/H for some closed subgp H.

Pf: Take $p \in M$. $G/G_p \xrightarrow{\sim} M$:

$$[g] \overset{\Phi}{\longmapsto} g \cdot p$$

$G_p = \{g : g \cdot p = p\}$ is closed $\Rightarrow$ G/G_p is a homog. G-mfd.

Φ is G-equiv. : $h \cdot \Phi([g]) = (h \cdot g) \cdot p = \Phi(h \cdot [g])$.

Φ is loc. diffeo. : $d\Phi|_{[1]}$ is inj. : if $d\Phi|_{[1]}([X]) = X_\#(p) = 0$, $[X] \in \mathfrak{g}/\mathfrak{g}_p$

$\Downarrow$ G-equiv.

$d\Phi|_{[g]}$ is inj. then p is fixed by $\exp tX \Rightarrow X \in \mathfrak{g}_p$.

$\Downarrow$

same dim. : $\dim G/G_p \le \dim M$.

If $<$, then $\exists\, q \in M$ st. $q \neq g \cdot p \;\forall g \longrightarrow \longleftarrow$ transitive.

Φ is bijective : $g \cdot p = h \cdot p \Rightarrow h^{-1} g \in G_p \Rightarrow [g] = [h]$.

Any $q = g \cdot p \in M$ for some g. #

21-14. Let V be an n-dimensional real vector space, and let $\mathrm{G}_k(V)$ be the Grassmannian of k-dimensional subspaces of V for some integer k with $0 < k < n$. Let $\mathbb{P}(\Lambda^k(V))$ denote the projectivization of $\Lambda^k(V)$ (see Problem 2-11). Define a map $\rho\colon \mathrm{G}_k(V) \to \mathbb{P}(\Lambda^k(V))$ by

$$\rho(S) = [v_1 \wedge \cdots \wedge v_k] \quad \text{if } S = \operatorname{span}(v_1, \ldots, v_k).$$

Show that ρ is well defined, and is a smooth embedding whose image is the set of all equivalence classes of nonzero decomposable elements of $\Lambda^k(V)$. (It is called the ***Plücker embedding***.)

21-18. The ***center*** of a group G is the set of all elements that commute with every element of G; a subgroup of G is said to be ***central*** if it is contained in the center of G. Show that every discrete normal subgroup of a connected Lie group is central. [Hint: use the result of Problem 7-8.]

$$\underset{\text{conn.}}{G} \xrightarrow{\text{conj}} H \text{ discrete} \Rightarrow \text{trivial action.}$$

21-19. Use the results of Theorem 7.7 and Problem 21-18 to show that the fundamental group of every Lie group is abelian. You may use without proof the fact that if $\pi\colon \widetilde{G} \to G$ is a universal covering map, then the automorphism group $\mathrm{Aut}_\pi(\widetilde{G})$ is isomorphic to $\pi_1(G, e)$ (see [LeeTM, Chap. 12]).

$$\underbrace{\pi_1(G)}_{\text{discrete}} = \underbrace{\pi^{-1}\{1\}}_{\text{normal}} \subset \widetilde{G} \text{ conn.}$$

$$\underbrace{\tilde{g}\,\underbrace{\gamma\,\underbrace{\tilde{g}^{-1}}_{\text{over } g^{-1}}}_{\text{fiber over } g^{-1}}}_{\text{fiber over } 1}$$