• Def: M admits a transitive action by a Lie group
• Examples: sphere = O(n)/O(n-1), upper half plane = SL(2,R)/SO(2), CP^n
  = U(n+1)/U(n)U(1), Gr(k,n) = U(n)/U(k)U(n-k) = GL(n)/U(n).
  Flag manifold = GL(n)/U(n) = U(n)/U(k).U(2) (k,2,...+n).
  Kähler
• G acts transitively on M <-> M = G/H for some closed subgroup H
• G acts transitively on set with closed isotropy subgp => S is a manifold.

• G/H is a homogeneous sp.  

  Pf.: Obviously it has a left transitive G-action.
  It is a mfd: G^\ast H freely: g\cdot h = g \Rightarrow h=1.
  Proper: g_i \mapsto g_i \Rightarrow h_i = g^i_\ast (g_i h_i) \mapsto g^{-1} \bar{g} \in H.
  G-action is sm.: G x G \xrightarrow{\varphi} G
  \xrightarrow{\pi_1} \xrightarrow{\pi_2 \circ \varphi} \xrightarrow{\pi_2} 
  G x (G/H) \xrightarrow{\varphi} G/H
  \pi_2 \circ \varphi sm. \Rightarrow \bar{\varphi} sm. since \pi_1 is a subm.
  (loc. word comes from upstairs) #
A homogeneous $G$-mfd $M$ is $\mathcal{G}/\mathcal{H}$ for some closed subgroup $\mathcal{H}$.

**Proof:** Take $p \in M$. $\mathcal{G}/\mathcal{G}_p \xrightarrow{\cong} M$:

$[g] \xrightarrow{\Phi} g \cdot p$

$G_p = \{g : g \cdot p = p\}$ is closed $\Rightarrow \mathcal{G}/\mathcal{G}_p$ is a homogeneous $G$-mfd.

$\Phi$ is $G$-equiv.: $h \cdot \Phi([g]) = (h \cdot g) \cdot p = \Phi(h \cdot [g])$.

$\Phi$ is local diffeo.: $d\Phi_{|[g]}$ is inj.: if $d\Phi_{|[g]}(\vec{X}) = \vec{X}_\#(p) = 0$,

$\downarrow G$-equiv.

$G/\mathcal{G}_p$

$d\Phi_{|[g]}$ is inj. then $p$ is fixed by $\exp_X \Rightarrow X \in \mathcal{G}_p$.

same dim.: $\dim \mathcal{G}/\mathcal{G}_p \leq \dim M$.

If $<$, then $\exists q \in M$ s.t. $q \neq g \cdot p \forall g \xrightarrow{\text{trans}}$.

$\Phi$ is bijective: $g \cdot p = h \cdot p \Rightarrow h^{-1}g \in G_p \Rightarrow [g] = [h]$.

Any $q = g \cdot p \in M$ for some $g$.
21-14. Let \( V \) be an \( n \)-dimensional real vector space, and let \( G_k(V) \) be the Grassmannian of \( k \)-dimensional subspaces of \( V \) for some integer \( k \) with \( 0 < k < n \). Let \( \mathbb{P}(\Lambda^k(V)) \) denote the projectivization of \( \Lambda^k(V) \) (see Problem 2-11). Define a map \( \rho: G_k(V) \to \mathbb{P}(\Lambda^k(V)) \) by
\[
\rho(S) = [v_1 \wedge \cdots \wedge v_k] \text{ if } S = \text{span}(v_1, \ldots, v_k).
\]
Show that \( \rho \) is well defined, and is a smooth embedding whose image is the set of all equivalence classes of nonzero decomposable elements of \( \Lambda^k(V) \). (It is called the \textit{Plücker embedding}.)

21-18. The \textit{center} of a group \( G \) is the set of all elements that commute with every element of \( G \), a subgroup of \( G \) is said to be \textit{central} if it is contained in the center of \( G \). Show that every discrete normal subgroup of a connected Lie group is central. [Hint: use the result of Problem 7-8.]

21-19. Use the results of Theorem 7.7 and Problem 21-18 to show that the fundamental group of every Lie group is abelian. You may use without proof the fact that if \( \pi: \tilde{G} \to G \) is a universal covering map, then the automorphism group \( \text{Aut}_\pi(\tilde{G}) \) is isomorphic to \( \pi_1(G, e) \) (see [LeeTM, Chap. 12]).