MATH 230A. Differential Geometry.

Lecture 10. Symplectic structure

[Diagram of a cube and water droplets, possibly related to symplectic geometry concepts.]
Motivation (Hamiltonian dynamics)

\[ X = \mathbb{R}^3 \] our space.

\[ M = T^* \mathbb{R}^3 = \mathbb{R}^3 \times (\mathbb{R}^3)^* \] 'phase space' of a particle.

\[ (q, p) \]
- position
- momentum

How can we talk about momentum without measuring the velocity?
Momentum of a particle at \( q \) assigns to each \( v \in T_q \mathbb{R}^3 \) the 'impact along \( v \) \( \in \mathbb{R} \). Hence it is a covector.

Newton's law: \[ F(q(t)) = m \dddot{q}(t). \]
2nd order ODE.

Written as 1st order ODE system:
\[
\begin{aligned}
q'(t) &= \frac{1}{m} p(t) \\
p'(t) &= F(q(t))
\end{aligned}
\]
Suppose the force field \( F \) have a 'source':

\[
\sum_i F_i(q) dq_i = -dW.
\]

called potential

eg. \( F \, dh = -g \, dh \).

\( W = gh \).

Take \( H(q,p) = \frac{1}{2m} ||p||^2 + W(q) \).

(Total energy, or Hamiltonian.)
$$dH = \sum_i \frac{p_i}{m} \, dp_i - \sum_i F_i(q) \, dq_i.$$  

$$X_H = \sum_i \frac{p_i}{m} \frac{\partial}{\partial q_i} + \sum_i F_i(q) \frac{\partial}{\partial p_i}.$$  

vector field  

\[ \omega = \sum_i dq_i \wedge dp_i. \]

\[ \therefore \quad \text{Newton's law} \]

\[ \leftrightarrow \quad \text{Hamiltonian flow} \quad \left( \frac{q}{p} \right)(t) = X_H \left( \left( \frac{q}{p} \right)(t) \right). \]

Want to formulate a global theory out of this structure.
Def.: \( V: \mathbb{R}\text{-vector space.} \)

\( \omega \in \Lambda^2 V^* \) is **non-degenerate** if

\( \exists \omega: V \to V^* \) is an isomorphism.

Then \((V, \omega)\) is called a **symplectic vector space**.

\[ M: \text{smooth manifold.} \]

Def.: \( \omega \in \Omega^2(M) \) is a **symplectic form** if

1. \((1) \forall p \in M, \omega_p \) is non-degenerate.
2. \((2) \omega \) is closed, i.e. \( d\omega = 0 \).

• If \( \omega \) is a symplectic form, 

\((M, \omega)\) is called a **symplectic manifold**.
Standard form for skew-symmetric pairing

\textbf{ex.} (1) $\forall \omega \in \Lambda^2 V^*$, $\exists$ basis \\
\{v_1, v_2, \ldots, v_r, w_1, \ldots, w_r, u_1, \ldots, u_k\} of \ V$ such that \\
(called symplectic basis if $k=0$)

$$\omega = \sum_{i=1}^{r} v_i^* \wedge w_i^*.$$ \\
($r$ is called the rank of $\omega$.)

(2) $\omega$ is non-deg. $\iff k = 0$ \\
$\iff \omega^r \neq 0 \in \Lambda^{2r}(V^*)$ \\
and dim $V = 2r$.

Thus \\
$\omega \in \Omega^2(M)$ non-degenerate \\
$\iff \dim M = 2n$, and $\omega^n |_p \neq 0 \ \forall p \in M.$ \\

\textcolor{red}{\color{red}{\text{volume form}}}

\textcolor{red}{\color{red}{\therefore \ A \ syplectic \ manifold \ is \ always \ even \ dimensional \ oriented.}}
Sketch of answer to e.x.:

If \( \omega = 0 \), take any basis of \( V \).

Otherwise \( \exists v_1 \in V \) such that \( \omega(v_1, w) \neq 0 \).

Then \( \exists w_1 \in V \) s.t. \( \omega(v_1, w_1) = 1 \).

Consider

\[
\{ v_1, w_1 \}^\perp_\omega = \{ v \in V : \omega(v, v_1) = \omega(v, w_1) = 0 \} \subset V.
\]

\[
V = (\text{Span } \{ v_1, w_1 \}) \oplus \{ v_1, w_1 \}^\perp_\omega.
\]

\begin{itemize}
  \item \( \text{Span } \{ v_1, w_1 \} \cap \{ v_1, w_1 \}^\perp_\omega = 0 \) since \( \omega(v_1, w_1) = 1 \).
  \item \( \forall v \in V, v = v_1 + v_2 \), where
    \[
    v_1 = \omega(v, w_2) v_2 - \omega(v, v_2) w_1 \in \text{Span } \{ v_1, w_1 \}.
    \]
    \[
    v_2 = v + \omega(v, v_2) w_1 - \omega(v, w_1) v_2 \in \{ v_1, w_1 \}^\perp_\omega.
    \]
\end{itemize}

Then continue the same procedure for \( \omega|_{\{ v_1, w_1 \}^\perp_\omega} \).
Analog (symmetric bilinear pairing):

For \( g \in \text{Sym}^2(V^*) \), \( \exists \) basis \( \{v_1, \ldots, v_r, w_1, \ldots, w_s, u, \ldots, u_b\} \) such that

\[
g = \sum_{i=1}^{r} v_i^* \otimes v_i^* - \sum_{j=1}^{s} w_j^* \otimes w_j^*.
\]

If \( r+s = \dim V \), \( g \) is called a metric (or inner product) of \((r,s)\) type.

\( g \in \text{Sym}^2(V^*) \) measures angles and lengths, and

\( w \in \Lambda^2 V^* \) measures area.
Symplectic volume

For $N \subset M$ of dim $= 2$, symplectic area of $N \triangleq \int_N i^* \omega$.

which only depends on homotopy class of $N$ by Stokes' theorem and $d\omega = 0$.

\[
\begin{align*}
\text{(If } N_1 - N_2 &= \emptyset, \text{ then } } \\
\int_{N_1 - N_2} i^* \omega &= \int_{\emptyset} i^* d\omega = 0.
\end{align*}
\]

May also consider $\int_N i^* \omega^k$ for $\dim N = 2k$.

ex. $[\omega] \neq 0 \in H^2(M)$ if $\omega$ symplectic, $M$ compact.

(consider $[\omega]^n$.)


Rmk: May replace \( \mathbb{R} \) by \( \mathbb{C} \).

\( V \): complex vector space. \( \omega \in \Lambda^2 V \) non-degenerate

\( \sim (V, \omega) \): complex symplectic vector space.

\( M \): complex manifold.

\( \omega \in \Lambda^2(T^*_0 M) \) closed, non-degenerate.

\( \omega = \sum_{i<j} w_{ij} \, dz_i \wedge d\bar{z}_j \) (locally)

\( \sim (M, \omega) \): complex symplectic manifold.
e.g. \((T^*X, \omega_{\text{can}})\) is a symplectic manifold.

Canonical 1-form:

For \(\eta \in T^*X\) and \(v \in T_\eta(T^*X)\),

\[\alpha_{\text{can}}(v) \triangleq \eta(d\pi(v)).\]

where \(\pi : T^*X \rightarrow X\) is the bundle map.

Canonical symplectic form:

\[\omega_{\text{can}} \triangleq -d\alpha_{\text{can}}.\]
ex. Let $q = (q_1, \ldots, q_n) : U \to \mathbb{R}^n$ be local coordinates

$\Rightarrow$ coordinates $T^* U \to \mathbb{R}^{2n}$

$\sum_{i=1}^{n} p_i dq_i \bigg|_x \mapsto (p_1, \ldots, p_n, q_1, \ldots, q_n)$.

Show that $\alpha_{can} \bigg|_{T^* U} = \sum_{i=1}^{n} p_i dq_i$.

$\therefore \omega_{can} = -d \alpha_{can} \bigg|_{T^* U} = \sum_{i=1}^{n} dq_i \wedge dp_i$.

eg. Any oriented surface is a symplectic manifold.
e.g. $\mathbb{C}P^n$ is a symplectic manifold: $\cup [z_o : \cdots : z_n]$ (indeed Kähler)

Let $\vec{z} = (\vec{z}_1, \cdots, \vec{z}_n) = (\frac{z_1}{\bar{z}_0}, \cdots, \frac{z_n}{\bar{z}_0})$ be inhomogeneous coordinates on $\{z_0 \neq 0\} \subset \mathbb{P}^n$.

**Fubini-Study form:**

$$
\omega \triangleq \frac{i}{2} \Theta \Theta \log ||\vec{z}||^2
$$

$$
= \frac{i}{2} \left( \frac{1}{1 + ||\vec{z}||^2} \sum_{i=1}^{n} d\vec{z}_i d\bar{z}_i - \frac{1}{(1 + ||\vec{z}||^2)^2} (\sum_{j} \vec{z}_j d\bar{z}_j) \wedge (\sum_{i} \vec{z}_i \bar{d}z_i) \right)
$$

ex. $\omega$ is $\mathbb{R}$-valued, $d\omega = 0$, and $\omega$ extends to be a symplectic form on $\mathbb{C}P^n$.

(Check that $\omega$ remains in the same form under change of inhomogeneous coordinates.)
e.g. All complex submanifolds of $\mathbb{CP}^n$ are symplectic. (Since $\omega(u, Jv) \neq 0 \ \forall u \neq 0$.)

The symplectic group. (Symmetries of symplectic structures)

$$\text{Sp}(2n, \mathbb{R}) \equiv \{ A \in \text{GL}(2n, \mathbb{R}) : A^* \omega_0 = \omega_0 \}$$

where $\omega_0 \equiv e_1^* \wedge f_1^* + e_2^* \wedge f_2^* + \ldots + e_n^* \wedge f_n^*$,

(standard symplectic form)

$\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is the standard basis of $\mathbb{R}^{2n}$.

In matrix, $\omega_0$ is represented by

$$\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}_n$$
\[ \text{ex.} \cdot \text{Sp}(2n, \mathbb{R}) \text{ is a manifold.} \]

\[ \{A^T(-I^I)A = (-I^I)\}. \]

\[ A^T(-I^I)A \in \{\text{skew-symmetric matrices}\} \]

\[ T \text{Sp}(2n, \mathbb{R}) = \overline{\text{Sp}(2n, \mathbb{R})}, \text{ where} \]

\[ \overline{\text{Sp}(2n, \mathbb{R})} = \left\{ A \in \text{Mat}_{2n}^\top(2n, 2n) : A^T(-I^I) = (-I^I)A \right\}. \]

\[ \sim \text{Sym}_{2n}^{\text{symmetric matrix}}(2n, 2n). \]

\[ \text{as vector space} \]

\[ \text{Sp}(2n, \mathbb{R}) \leq \text{SL}(2n, \mathbb{R}). \]

(Because \( w_0^n \) gives the standard volume form up to \( \pm \).)
A symplectic structure on a vector bundle $E \rightarrow M$ is a map $\omega \in \Gamma(\Lambda^2 E^*)$ s.t. $\omega_p$ is non-degenerate $\forall p \in M$.

**e.g.** $TM$ of a symplectic manifold $(M,\omega)$ is a symplectic vector bundle.

**ex.** $\forall p \in M$, $\exists \cup U \subseteq M$ and local symplectic frame

$\{v_i, w_i\}_{i=1}^r \subseteq \Gamma(U, E)$ s.t. $\omega|_U = \sum_{i=1}^r v_i^* \wedge w_i^*$.

(Do the same thing as picking symplectic basis.)

**Therefore** $\exists$ local symplectic trivialization around every $p \in M$. 
Reduction of structure group

For a symplectic vector bundle:

A local trivialization $E|_u \xrightarrow{\varphi} U \times \mathbb{R}^{2r}$ is symplectic if

$$(\varphi^{-1})^* \omega|_u = \sum_{i=1}^{r} e_i^* \wedge f_i^*, \quad \gamma^{-1} f_i^* e_i$$

where $\{e_i, f_i\}$ is the standard basis on $\mathbb{R}^{2r}$.

By collecting all local symplectic trivializations, the transition $g_{ji} : U_{ji} \rightarrow GL(2n, \mathbb{R})$ has image in $Sp(2n, \mathbb{R})$.

\[ \because \text{For symplectic manifold } (M, \omega), \text{ structure group of } TM \text{ is reduced to } Sp(2n, \mathbb{R}). \]
Symplectic structure is always locally trivial: Darboux theorem:

\( \forall p \in M, \exists \text{ chart } (U, (q,p)) \) such that

\[
\omega \big|_U = \sum_{i=1}^{n} dq_i \wedge dp_i.
\]

i.e. \( \exists \) local symplectic coordinate frame.

Need \( \{d\omega = 0\} \)!

For Riemannian metric \( g \in \text{Sym}^2(T^*M) \), the condition \( 'dg=0' \) does not make sense.

In general \( \# \) local orthonormal coordinate frame.

Curvature: a local invariant to capture such non-triviality.