# MATH 230A Differential Geometry.

## Lecture 10. Symplectic structure

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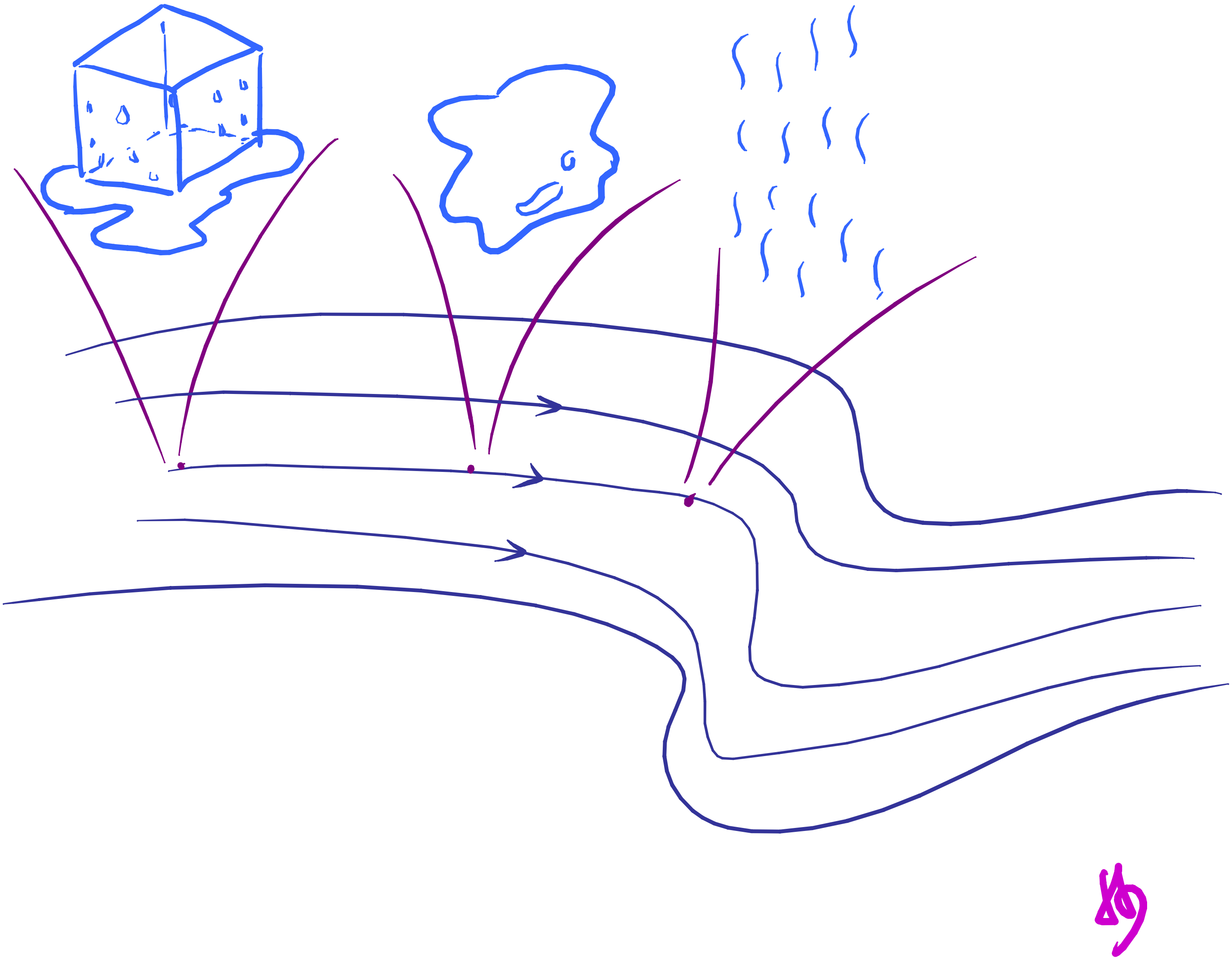
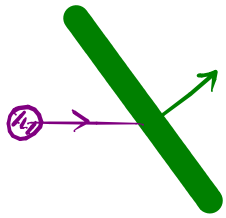


# Motivation (Hamiltonian dynamics)

$X = \mathbb{R}^3$ our space.

$M = T^*\mathbb{R}^3 = \mathbb{R}^3 \times (\mathbb{R}^3)^* \ni (\underbrace{q}_{\text{position}}, \underbrace{p}_{\text{momentum}})$ 'phase space' of a particle.

How can we talk about momentum without measuring the velocity? Momentum of a particle at $q$ assigns to each $v \in T_q\mathbb{R}^3$ the 'impact along $v$' $\in \mathbb{R}$. Hence it is a covector.



measure the 'impact' (i.e. kinetic energy)

Newton's law: $\overbrace{F(q(t))}^{\text{vector field}} = \overbrace{m}^{\text{constant}} q''(t)$.

$2^{nd}$ order ODE.

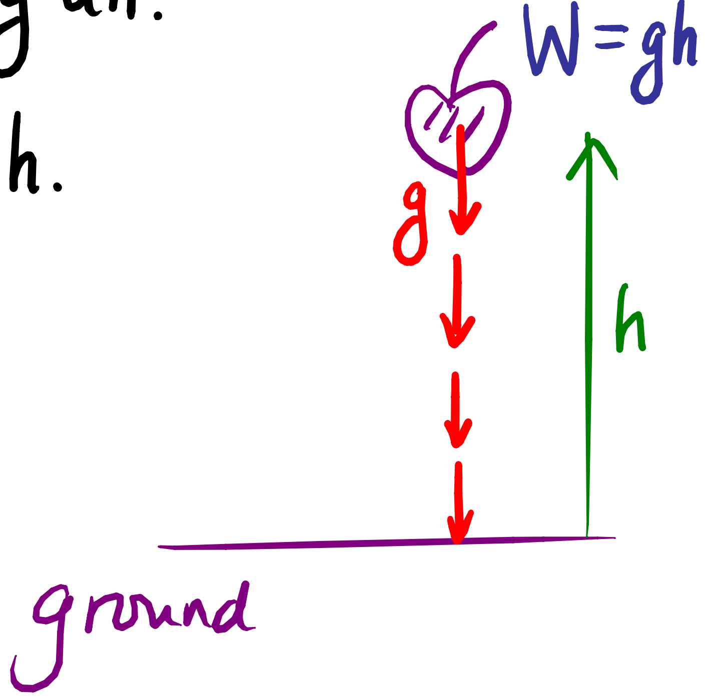
written as $1^{st}$ order ODE system:
$$\begin{cases} q'(t) = \frac{1}{m} p(t). \\ p'(t) = F(q(t)). \end{cases}$$

Suppose the force field $F$ have a 'source':

$$\sum_i F_i(q)\, dq_i = -d\underbrace{W}_{\text{called potential}}.$$

e.g. $F\, dh = -g\, dh.$

$$W = gh.$$




Take $H(q,p) = \dfrac{\|p\|^2}{2m} + W(q)$.

(Total energy, or Hamiltonian.)

$$dH = \sum_i \frac{p_i}{m}\, dp_i - \sum_i F_i(q)\, dq_i .$$

$$\xrightarrow[\text{LW}]{} \underbrace{X_H}_{\text{vector field}} = \sum_i \frac{p_i}{m}\frac{\partial}{\partial q_i} + \sum_i F_i(q)\frac{\partial}{\partial p_i} .$$

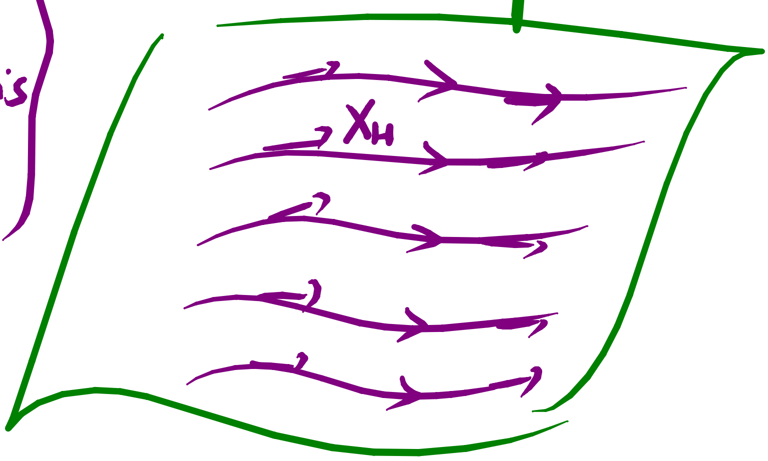
setting $\boxed{\omega = \sum_i dq_i \wedge dp_i .}$

$\therefore$ Newton's law

$\Longleftrightarrow$ Hamiltonian flow $\begin{pmatrix} q \\ p \end{pmatrix}'(t) = X_H\left(\begin{pmatrix} q \\ p \end{pmatrix}(t)\right).$

Want to formulate a global theory out of this structure.

Phase space



Def. : $V$: $\mathbb{R}$-vector space.

$\omega \in \Lambda^2 V^*$ is non-degenerate if

$\iota . \omega : V \longrightarrow V^*$ is an isomorphism.

Then $(V, \omega)$ is called a symplectic vector space.

$M$ : smooth manifold.

Def. : • $\omega \in \Omega^2(M)$ is a symplectic form if

pointwise condition (1) $\forall p \in M$, $\omega|_p$ is non-degenerate.

'constant' condition (2) $\omega$ is closed, i.e. $d\omega = 0$.

• If $\omega$ is a symplectic form, $(M, \omega)$ is called a symplectic manifold.

# Standard form for skew-symmetric pairing

ex. (1.) $\forall\, \omega \in \Lambda^2 V^*$, $\exists$ basis
$\{v_1, v_2, \dots, v_r, w_1, \dots, w_r, u_1, \dots u_k\}$ of $V$ such that
(called symplectic basis if $k=0$.)

$$\omega = \sum_{i=1}^{r} v_i^* \wedge w_i^* .$$

($r$ is called the rank of $\omega$.)

(2.) $\omega$ is non-deg. $\Leftrightarrow k = 0$

$\Leftrightarrow \omega^r \neq 0 \in \Lambda^{2r}(V^*)$ and $\dim V = 2r$.

Thus

$\omega \in \Omega^2(M)$ non-degenerate

$\Leftrightarrow \dim M = 2n$, and $\underbrace{\omega^n|_p}_{\text{volume form}} \neq 0 \ \forall p \in M$.

$\therefore$ A symplectic manifold is always even dimensional oriented.

Sketch of answer to e.x.:

If $\omega = 0$, take any basis of $V$.

Otherwise $\exists v_1 \in V$ such that $\iota_{v_1}\omega \neq 0$.

Then $\exists w_1 \in V$ s.t. $\omega(v_1, w_1) = 1$.

Consider

$$\{v_1, w_1\}^{\perp\omega} \triangleq \{v \in V : \omega(v, v_1) = \omega(v, w_1) = 0\} \subset V.$$

$$V = \left(\operatorname{Span}\{v_1, w_1\}\right) \oplus \{v_1, w_1\}^{\perp\omega}:$$

- $\operatorname{Span}\{v_1, w_1\} \cap \{v_1, w_1\}^{\perp\omega} = 0$ since $\omega(v_1, w_1) = 1$.
- $\forall v \in V$, $v = v_1 + v_2$, where

$$v_1 = \omega(v, w_1)v_1 - \omega(v, v_1)w_1 \in \operatorname{Span}\{v_1, w_1\}.$$

$$v_2 = v + \omega(v, v_1)w_1 - \omega(v, w_1)v_1 \in \{v_1, w_1\}^{\perp\omega}.$$

Then continue the same procedure for

$$\omega\big|_{\{v_1, w_1\}^{\perp\omega}}.$$

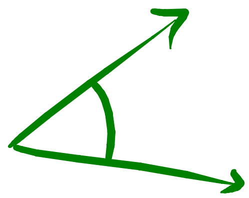
Analog (symmetric bilinear pairing):

ex. For $g \in \mathrm{Sym}^2(V^*)$, $\exists$ basis $\{v_1, \dots, v_r, w_1, \dots, w_s, u_1 \dots u_k\}$ such that

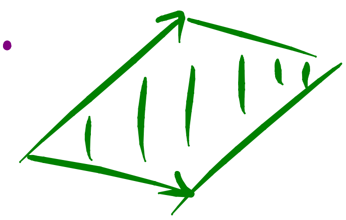
$$g = \sum_{i=1}^{r} v_i^* \otimes v_i^* - \sum_{j=1}^{s} w_j^* \otimes w_j^* .$$

If $r+s = \dim V$, $g$ is called a metric (or inner product) of $(r,s)$ type.

$g \in \mathrm{Sym}^2(V^*)$ measures angles and lengths, and



$\omega \in \Lambda^2 V^*$ measures area.



# Symplectic volume

For $\underset{\text{submanifold}}{N \subset M}$ of $\dim = 2$,

symplectic area of $N \triangleq \int_N \iota^* \omega$.

which only depends on homotopy class of $N$ by Stokes' theorem and $d\omega = 0$.

(If $N_1 - N_2 = \partial P$,
then $\int_{N_1 - N_2} \iota^* \omega = \int_P \iota^* \cancel{d\omega}^{\,0} = 0$.)

May also consider $\int_N \iota^* \omega^k$ for $\dim N = 2k$.

ex. $[\omega] \neq 0 \in H^2(M)$ if $\omega$ symplectic, $M$ compact.
(consider $[\omega]^n$.)

Rmk: May replace $\mathbb{R}$ by $\mathbb{C}$.

$V$ : complex vector space. $\omega \in \Lambda^2 V$ non-degenerate

$\rightsquigarrow (V, \omega)$ : complex symplectic vector space.

$M$ : complex manifold.

$\omega \in \Lambda^2(T^*_{1,0} M)$ closed, non-degenerate.

$$\left( \omega \underset{\text{locally}}{=} \sum_{i<j} \omega_{ij} \, dz_i \wedge dz_j \right)$$

$\rightsquigarrow (M, \omega)$: complex symplectic manifold.

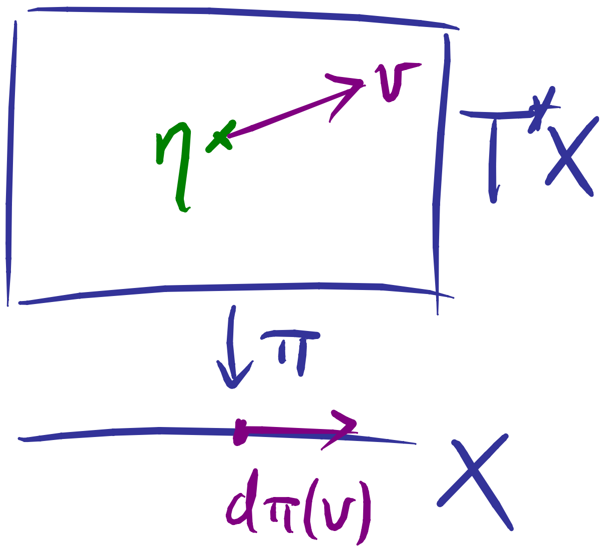
e.g. $(T^*X, \omega_{can})$ is a symplectic manifold.

Canonical 1-form:

For $\eta \in T^*X$ and $v \in T_\eta(T^*X)$,

$$\alpha_{can}(v) \triangleq \eta(d\pi(v)).$$

where $\pi : T^*X \longrightarrow X$ is the bundle map.




Canonical symplectic form:

$$\omega_{can} \triangleq -d\alpha_{can}.$$

ex. Let $q = (q_1, \dots, q_n) : \underset{\substack{\cap \text{open} \\ X}}{U} \longrightarrow \mathbb{R}^n$

be local coordinates

$$\Rightarrow \text{coordinates } \begin{array}{ccc} T^*U & \longrightarrow & \mathbb{R}^{2n} \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ \sum_{i=1}^{n} p_i \, dq_i \Big|_x & \longmapsto & (p_1, \dots, p_n, q_1, \dots, q_n). \end{array}$$

Show that $\alpha_{can}\Big|_{T^*U} = \sum_{i=1}^{n} p_i \, dq_i.$

$$\therefore \ \omega_{can} = -d\alpha_{can}\Big|_{T^*U} = \sum_{i=1}^{n} dq_i \wedge dp_i .$$

e.g. Any oriented surface is a symplectic manifold.

e.g. $\mathbb{C}P^n$ is a symplectic manifold:
$\overset{\Psi}{[z_0 : \dots : z_n]}$ (indeed Kähler)

Let $\vec{\zeta} \triangleq (\zeta_1, \dots, \zeta_n) = \left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right)$ be inhomogeneous coordinates on $\{z_0 \neq 0\} \subset \mathbb{P}^n$.

Fubini-Study form:

$$\omega \triangleq \frac{i}{2} \partial \bar{\partial} \log \|\vec{z}\|^2$$

$$= \frac{i}{2}\left(\frac{1}{1+\|\vec{\zeta}\|^2} \sum_{i=1}^{n} d\zeta_i \wedge d\bar{\zeta}_i - \frac{1}{(1+\|\vec{\zeta}\|^2)^2}\left(\sum_j \bar{\zeta}_j d\zeta_j\right) \wedge \left(\sum_i \zeta_i d\bar{\zeta}_i\right)\right).$$

e.x. $\omega$ is $\mathbb{R}$-valued, $d\omega = 0$, and

$\omega$ extends to be a symplectic form on $\mathbb{C}P^n$.

(Check that $\omega$ remains in the same form under change of inhomogeneous coordinates.)

e.g. All complex submanifolds of $\mathbb{CP}^n$ are symplectic. (Since $\omega(u, Ju) \neq 0 \; \forall u \neq 0$.)

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## The symplectic group. (Symmetries of symplectic structures)

$$Sp(2n, \mathbb{R}) \triangleq \{A \in GL(2n, \mathbb{R}) : A^* \omega_0 = \omega_0\}$$

where $\omega_0 \triangleq e_1^* \wedge f_1^* + e_2^* \wedge f_2^* + \dots + e_n^* \wedge f_n^*$,
(standard symplectic form)

$\{e_1, \dots, e_n, f_1, \dots, f_n\}$ is the standard basis of $\mathbb{R}^{2n}$.

In matrix, $\omega_0$ is represented by
$$\begin{array}{cc} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} & \begin{array}{l} ]\,n \\ ]\,n \end{array} \\ \begin{array}{cc} \underbrace{\phantom{-I}}_{n} & \underbrace{\phantom{0}}_{n} \end{array} & \end{array}.$$

ex. $\cdot\ Sp(2n,\mathbb{R})$ is a manifold.

$$\begin{aligned} Sp(2n,\mathbb{R}) &= \left\{A^T\begin{pmatrix} & I \\ -I & \end{pmatrix}A = \begin{pmatrix} & I \\ -I & \end{pmatrix}\right\}. \end{aligned}$$

$A^T\begin{pmatrix} & I \\ -I & \end{pmatrix}A \in \{\text{skew-symmetric matrices}\}$

$\cdot\ T\,Sp(2n,\mathbb{R}) = \underline{\mathfrak{sp}(2n,\mathbb{R})}$, where

$$\mathfrak{sp}(2n,\mathbb{R}) \triangleq \left\{ a \in Mat_{\mathbb{R}}(2n,2n) : a^T\begin{pmatrix} & I \\ -I & \end{pmatrix} = \underbrace{\begin{pmatrix} & -I \\ I & \end{pmatrix} a}_{\text{symmetric matrix}} \right\}.$$

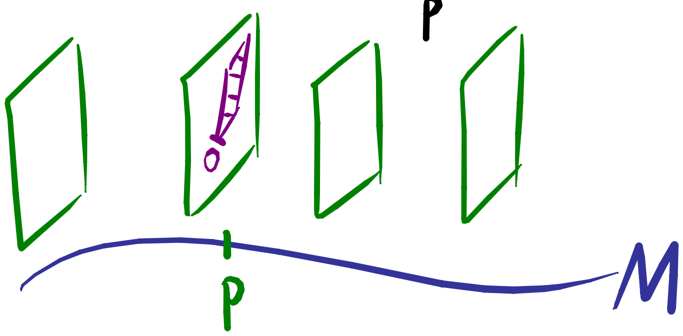
$$\underset{\text{as vector space}}{\cong} Sym_{\mathbb{R}}(2n,2n).$$

$\cdot\ Sp(2n,\mathbb{R}) \leqslant SL(2n,\mathbb{R})$

(Because $\omega_0^n$ gives the standard volume form up to $\pm$).

A <u>symplectic structure on a vector bundle $E \to M$</u> is

$\omega \in \Gamma(\Lambda^2 E^*)$ s.t. $\omega|_p$ is non-degenerate $\forall p \in M$.




e.g. $TM$ of a symplectic manifold $(M, \omega)$ is a symplectic vector bundle.

ex. $\forall p \in M$, $\exists \underset{\substack{\cup \\ p}}{U} \underset{\text{open}}{\subset} M$ and local symplectic frame

$$\{v_i, w_i\}_{i=1}^{r} \subset \Gamma(U, E) \text{ s.t. } \omega|_U = \sum_{i=1}^{r} v_i^* \wedge w_i^* .$$

(Do the same thing as picking symplectic basis.)

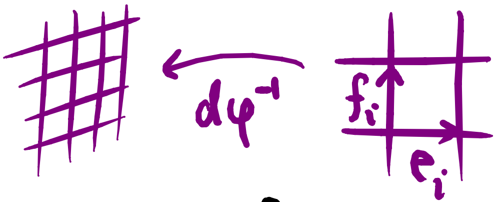
$\therefore$ $\exists$ local symplectic trivialization around every $p \in M$.

# Reduction of structure group

For a symplectic vector bundle:

a local trivialization $E|_U \xrightarrow{\varphi} U \times \mathbb{R}^{2r}$ is symplectic if

$$(\varphi^{-1})^* \omega|_U = \sum_{i=1}^{r} e_i^* \wedge f_i^*,$$




where $\{e_i, f_i\}$ is the standard basis on $\mathbb{R}^{2r}$.

By collecting all local symplectic trivializations, the transition $g_{ji} : U_{ji} \longrightarrow GL(2n,\mathbb{R})$ has image in $Sp(2n,\mathbb{R})$.

$\therefore$ For symplectic manifold $(M,\omega)$, structure group of $TM$ is reduced to $Sp(2n,\mathbb{R})$.

Symplectic structure is always locally trivial:

Darboux theorem:

$\forall p \in M, \exists$ chart $(\mathcal{U}, (q,p))$ such that

$$\omega|_{\mathcal{U}} = \sum_{i=1}^{n} dq_i \wedge dp_i .$$

i.e. $\exists$ local symplectic coordinate frame.

Need $\boxed{d\omega = 0}$ !

For Riemannian metric $g \in \mathrm{Sym}^2(T^*M)$, the condition '$dg = 0$' does not make sense.

In general $\nexists$ local orthonormal coordinate frame.

Curvature: a local invariant to capture such non-triviality.