MATH 230A. Differential Geometry

Lecture 11. Lagrangians and symplectomorphisms

ref.: [da Silva Ch.3,4]

BRANE WORLD.....
Defn: \((V, \omega)\): symplectic vector space.

\[ W \subseteq V \] is

- isotropic if \(W \subseteq W^\perp \omega\).
- Lagrangian if \(W = W^\perp \omega\).
- coisotropic if \(W \supseteq W^\perp \omega\).
- symplectic if \(\omega|_W\) is non-degenerate.

(Recall \(W^\perp \omega = \{v \in V : \omega(v, w) = 0 \ \forall w \in W\}\) )
\textbf{ex.} \quad \dim W + \dim W^\perp = \dim V.

Consider $\quad V \xrightarrow{\iota, \omega} V^* \rightarrow W^*.$

\textbullet \quad \text{Ker} = W^\perp :

$v \in \text{Ker} \iff (\iota, \omega)(v)(w) = \omega(v, w) = 0 \quad \forall w \in W

\iff v \in W^\perp.

\textbullet \quad \text{Im} = W^* :

\forall w \in W, \exists v \in V \text{ such that } \omega(v, w) \neq 0.

\textbullet \quad \ker + \text{rank} = n.
\[(W^\perp)^\perp = W, \quad (W \text{ is non-degenerate})\]

- \[W \text{ symplectic} \iff W \cap W^\perp = 0 \iff V = W \oplus W^\perp.\]

Use \[\dim W + \dim W^\perp = V.\]

- \[W \text{ isotropic} \implies \dim W \leq \frac{1}{2} \dim V.\]

- \[W \text{ coisotropic} \implies \dim W \geq \frac{1}{2} \dim V.\]

- \[W \text{ Lagrangian} \iff W \text{ isotropic and} \quad \dim W = \frac{1}{2} \dim V.\]
If $W \subseteq (V, \omega)$ is isotropic, any basis of $W$ can be extended to a symplectic basis of $V$.

For a basis $\{e_1, \ldots, e_k\}$ of $W$

If $W \not\subseteq W^\perp$, pick $e_{k+1} \in W^\perp - W$.

If $\text{Span} \{e_1, \ldots, e_{k+1}\} \not\subseteq \{e_1, \ldots, e_{k+1}\}^\perp$

pick $e_{k+2} \in \{e_1, \ldots, e_{k+1}\}^\perp - \text{Span} \{e_1, \ldots, e_{k+1}\}$.

Inductively get $\{e_1, \ldots, e_n\} = \{e_1, \ldots, e_n\}^\perp$.

\forall j, \text{ Pick } f_j \in \{e_1, \ldots, \hat{e_j}, \ldots, e_n\}^{\perp_{\text{dim.}}} - W \text{ s.t. } \omega(e_j, f_j) = 1.

\left( W \not\subseteq \{e_1, \ldots, \hat{e_j}, \ldots, e_n\}^{\perp_{\text{dim.}}} \right)
e.x. • If $W \subset V$ is Lagrangian, there exists a symplectomorphism

$$\Phi: (V, \omega) \sim (W \oplus W^*, \omega_{can})$$

(i.e. $\Phi^* \omega_{can} = \omega$)

with $\Phi(w) = (w, 0)$ for all $w \in W$.

• Extend a basis $\{e_i\}$ of $W$ to a symplectic basis $\{e_i, f_i\}$.

• $\Phi(e_i) \triangleq (e_i, 0)$ ;

• $\Phi(f_i) \triangleq (0, e_i^*)$

where $\{e_i^*\}$ is dual basis of $e_i$. 
ex. For $W \subset V$, \exists symplectic basis $\{e_i, f_i\}$ s.t.

$$W = \text{Span} \{e_1, \ldots, e_n, f_1, \ldots, f_k\}.$$  

In particular, if $W \subset V$,

\exists symplectic basis $\{e_i, f_i\}_{i=1}^n$ of $V$ such that

$$W = \text{Span} \{e_1, \ldots, e_k, f_1, \ldots, f_k\}.$$
Easy way to remember: (follow from above)

- \( W \) isotropic \( \iff \exists \) symplectic basis \( \{e_i, f_i\} \) s.t.
  \[
  W = \text{Span} \{ e_1, \ldots, e_n \}. \]

- \( W \) Lagrangian \( \iff \exists \) symplectic basis \( \{e_i, f_i\} \) s.t.
  \[
  W = \text{Span} \{ e_1, \ldots, e_n \}, \quad n = \frac{\dim M}{2}. \]

- \( W \) coisotropic \( \iff \exists \) symplectic basis \( \{e_i, f_i\} \) s.t.
  \[
  W = \text{Span} \{ e_1, \ldots, e_n, f_1, \ldots, f_k \}, \quad n = \frac{\dim M}{2}. \]

- \( W \) symplectic \( \iff \exists \) symplectic basis \( \{e_i, f_i\} \) s.t.
  \[
  W = \text{Span} \{ e_1, \ldots, e_k, f_2, \ldots, f_n \}. \]
Defn.: $(M, \omega)$ symplectic.

$X \subset M$ is Lagrangian/isotropic/coisotropic/symplectic

if $T_x X \subset T_x M$ is.

ex. 22-8. Suppose $(M, \omega)$ is a symplectic manifold and $S \subset M$ is a coisotropic submanifold. An immersed submanifold $N \subset S$ is said to be characteristic if $T_p N = (T_p S)^\perp$ for each $p \in N$. Show that there is a foliation of $S$ by connected characteristic submanifolds of $S$ whose dimension is equal to the codimension of $S$ in $M$.

Lagrangian submanifolds are considered as
the most important class in symplectic geometry.

- $X \subset T^* X$ is a Lagrangian submanifold.

- $T_q X \subset T^* X$ is a Lagrangian submanifold.
• 'branes' in string theory.
  (T-conditions for open string)

(Some string theorists claim that one should also consider 'coisotropic branes'.)

• Lagrangian intersection theory well-developed.

• Related to symplectomorphisms (see below).
Examples of Lagrangian submanifolds.

E.g. Any curve $\gamma$ in an oriented surface is a Lagrangian submanifold.

($\therefore \omega (\dot{\gamma}, \dot{\gamma}) = 0$)

E.x. (Lagrangian sections)

For $(T^*X, \omega_{can})$ and $\varphi \in \Gamma (T^*X)$,

$\text{gr}(\varphi) \subset T^*X \iff d\varphi = 0$.

$\omega_{can} = \sum_{i=1}^{n} dq_i \wedge dp_i$.

$\varphi = \sum_{i=1}^{n} \varphi_i \, dq_i$.

$\text{gr}(\varphi) = \{(q, \varphi(q))\}$.

$\varphi^* \omega_{can} = \sum_{i=0}^{n} dq_i \wedge dp_i = -d\varphi$. 
For \( S \subseteq X \), (conormal bundle of \( S \))
\[
N^*S = \{ v \in T^*X : \pi(v) \in S, \ v(v) = 0 \ \forall v \in TS \}.
\]
\[
\subseteq T^*X.
\]

\text{eg. } S = \{ p \} \Rightarrow N^*S = T^*_pX.

\text{eg. } S = X \Rightarrow N^*S = 0\text{-section of } T^*_pX.

\text{ex. } N^*S \subseteq T^*X \text{ is a Lagrangian submanifold.}

Let \( \{ q_i \}_{i=1}^n \) be local coordinates of \( X \) s.t.
\[ S = \{ q_i = 0 \ \forall i = k+1, \ldots, n \}. \quad (k = \dim S) \]
\[
N^*S = \{ \sum_{i=1}^n p_i dq_i \in T^*X : p_i = 0 \ \forall i = 1, \ldots, k \}.
\]

\[
\omega \left( \frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_j} \right) = 0 \quad \text{for } i = k+1, \ldots, n, \ j = 1, \ldots, k.
\]
e.g. \( \mathbb{C}^2 \), \( \omega_{std} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \).

\[
\begin{pmatrix}
  z_1^1 \\
  z_1^2 \\
  z_2^1 \\
  z_2^2
\end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ 0 \\ x_2 + iy_2 \\ 0 \end{pmatrix}
\]

ex. \( T_{c_1, c_2} \doteq \{ |z_i| = c_i \ \forall i = 1, 2 \}, c_1, c_2 \neq 0 \),

is a Lagrangian submanifold of \( \mathbb{C}^2 \).

For \( c_1 = 0 \) or \( c_2 = 0 \),

it is an isotropic submanifold of \( \mathbb{C}^2 \).

ex. \( \{ |z_1| = |w|, |z_2 - \overline{w}| = c \}, c \neq 0 \),

is a Lagrangian submanifold of \( \mathbb{C}^2 \).

\( c > 0 : \text{product tori.} \)

\( c < 0 : \text{Chekanov tori.} \)

\( c = 0 : \text{singular torus.} \)
**Symplectomorphism**

\((M_i, \omega_i)\) symplectic manifolds, \(i = 1, 2\).

**Defn.** \(\varphi : (M_1, \omega_1) \rightarrow (M_2, \omega_2)\) is **symplectic** if \(\varphi^*\omega_2 = \omega_1\).

It is a **symplectomorphism** if in addition it is a diffeomorphism.

\[ \text{Sympl}(M, \omega) \triangleq \{ \text{symplectomorphisms} \, (M, \omega), \mathcal{C} \} \]

is a subgroup of \(\text{Diffeo}\).
Symplectic vector field

For a vector field $v \in T^1(TM)$, (M compact)
It generates symplectomorphisms
i.e. $(\exp tv)^* \omega = \omega$

$\iff \frac{d}{dt} ((\exp tv)^* \omega) = 0.$

$\iff (\exp tv)^* L_v \omega$

$\iff L_v \omega = 0.$

\textbf{Defn} $v \in T^1(TM)$ is a \underline{symplectic vector field} if $L_v \omega = 0.$
\[ T_{\omega}(TM) \triangleq \{ \text{Symplectic vector fields} \} \subseteq \Gamma(TM) \]

is closed under Lie bracket.

(Use Leibniz rule)

\[ \mathcal{L}_v(\mathcal{L}_w \omega) = \mathcal{L}_{[v,w]} \omega + \mathcal{L}_v \mathcal{L}_w \omega \]

\[ T_{\omega}(TM) \] is ‘Lie algebra’ of Sympl\((M,\omega)\).
Hamiltonian vector fields

Def.: \( \nu \in \Gamma(TM) \) is a Hamiltonian vector field

\[ \nu = X_H \overset{\Delta}{=} (dH)_{\#_w} \text{ for some } H \in C^\infty(M). \]

(TM \overset{\#_w}{\leftrightarrow} T^*M)

\( \varphi \) is a Hamiltonian diffeomorphism if

\( \varphi = \exp t \nu \) for some Hamiltonian vector field \( \nu \), \( t \in \mathbb{R} \).

(Conservation of energy)

- A Hamiltonian vector field \( \nu \) is symplectic.

\[ L_\nu \omega = dL_\nu \omega = ddH = 0. \]

- It preserves the Hamiltonian \( H \):

\[ L_\nu H = 0. \]

\[ L_\nu H = dH(\nu) = \omega(\nu, \nu) = 0. \]
Poisson structure.

ex. {Hamiltonian vector fields} is closed under Lie bracket.

\[ [X_f, X_g] = X_{\{f, g\}} , \text{where } \{f, g\} \text{ is defined below.} \]

Defn. The Poisson structure induced by \( \omega \) is \( \{ \cdot, \cdot \} : \mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(M) \) defined by \( \{f, g\} \equiv \omega(X_f, X_g) \).

ex. \( \{f, g\} = X_g f = -X_f g \).

* Jacobi identity: \( \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \).
Noether principle

Fix \((M, \omega, H)\): Hamiltonian system. \((M\ \text{connected})\)

**Theorem:** Assume \(H^2(M) = 0\).

\[
\{\text{Conserved quantities}\} \overset{1-1}{\underset{\text{up to adding}}{\leftrightarrow}} \{\text{symmetries}\}
\]

\[
f: M \rightarrow \mathbb{R} \text{ s.t. } X_H \cdot f = 0
\]

\[
Pf: C^\infty(M) \overset{\sim}{\underset{\text{Ham. v.f.}}{\leftrightarrow}} H^1(M, \omega) \overset{\text{v.f.}}{\leftrightarrow} v.f. (M, \omega)
\]

\[
f \leftrightarrow X_f
\]

\[
X_H \cdot f = 0 \iff \{H, f\} = 0 \iff X_f \cdot H = 0.
\]
Hamiltonian flowout

\[ T \subseteq (M, w) \text{ s.t. } \begin{cases} T \subseteq H^{-1}\{c\} \\ X_H \text{ nowhere tangent to } T. \end{cases} \]

The flowout \( S \) from \( T \) along \( X_H \) is isotropic and \( S \subseteq H^{-1}\{c\} \).

\[ \begin{array}{c}
\text{Pf: } \omega(X_H, v) = dH(v) = 0 \quad \forall \ v \in T_T \\
\Rightarrow S \text{ is isotropic.} \\
X_H \cdot H = \{X_H, X_H\} = 0 \\
\Rightarrow H \big|_{\text{flowline of } X_H \text{ for } p \in T} = c \\
\Rightarrow H(S) = c. \ #
\end{array} \]
Lagrangians and symplectomorphisms.

From \((M_i, \omega_i)\) symplectic manifolds, \(i = 1, 2\), get \((M_1 \times M_2, \omega_\pm = \omega_1 \pm \omega_2)\) symplectic manifolds.

**e.x.** Let \(\varphi : M_1 \rightarrow M_2\) diffeomorphism.

\(\varphi\) is symplectomorphism

\(\Leftrightarrow \text{gr}(\varphi) \subset (M_1 \times M_2, \omega_-)\) is Lagrangian.

\(\because \omega_\pm|_{\text{gr}(\varphi)} = \omega_1 - \varphi^* \omega_2.\)
Cook up Lagrangians in \((M \times M_2, \omega_-)\)

Focus on \(M_i = (T^*X_i, \omega_{\text{can}})\).

ex. \((M_1 \times M_2, \omega_+) \cong (T^*(X_1 \times X_2), \omega_{\text{can}})\).

• Can cook up Lagrangian \( \subset (T^*(X_1 \times X_2), \omega_{\text{can}}) \)

by \(\text{graph}(\omega) \subset T^*X\), where

\(\omega \in \Omega^1_{\text{can}}(X)\).

e.g. take \(\omega = dH, H: X \rightarrow \mathbb{R}\).

• How about Lagrangian \( \subset (M_1 \times M_2, \omega_-) \)?

Want to have \( \sigma: M_1 \times M_2 \rightarrow \mathcal{X} \)

st. \(\sigma^* \omega_- = \omega_+\).

Then \(L \subset (T^*X, \omega_+) \Leftrightarrow \sigma(L) \subset (T^*X, \omega_-)\).
Involution on $T^*X$.

Ex. $\sigma : T^*X \ni v \mapsto -v$ has the property

$$\sigma^* \omega_{\text{can}} = -\omega_{\text{can}}.$$ 

$\sigma : \text{odd}$

$$\therefore (\text{Id}, \sigma) : T^*X_1 \times T^*X_2 \ni (v_1, v_2) \mapsto (v_1, -v_2) \implies \sigma \circ \sigma = \text{id}$$

$$\therefore \sigma^* \omega_- = \omega_+.$$ 

$$\therefore L \subseteq (T^*X, \omega_+) \iff \sigma(L) \subseteq (T^*X, \omega_-).$$

Then $H \sim L_H \triangleq \text{gr}(dH) \subseteq (T^*X, \omega_+)$

$$\sigma(L_H) \subseteq (T^*X, \omega_-).$$
Discrete version of Hamiltonian dynamics.

\[
\sigma(L_H)_{\text{lag.}}(T^*X, \omega_-) = \text{gr} (\varphi : T^*X_1 \to T^*X_2)
\]
(global condition)

\[
\Rightarrow \vartheta_x \vartheta_x \mid H : T^*_x X_1 \otimes T^*_x X_2 \to \mathbb{R}
\]
(local condition) non-degenerate \( \forall (x_1, x_2) \in X_1 \times X_2 \).

\[
\forall (\xi_x, \eta_y) \in T^*_x X_1 \times T^*_y X_2,
\]
given \( \xi_x \), solve for \( y \):

\[
\eta_y = \varphi(\xi_x) \iff \begin{cases}
\xi_x = d_x H(x, y) \\
\eta_y = -d_y H(x, y)
\end{cases}
\]

and hence show that

\[
\det \left( \partial_{x_i} \partial_{y_j} H \right) \neq 0 \text{ by taking derivative on RHS and use } \varphi \text{ is a diffeomorphism.}
\]
e.g. \( X_1 = X_2 = \mathbb{R}^n \).

Phase space (discrete version):

\[ \mathbb{R}^n \times \mathbb{R}^n. \]

Hamiltonian (discrete version):

\[ H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \]

\[ H(x, y) = -\frac{1}{2} |x - y|^2. \]

\[ = \sum_i \frac{1}{2} (x_i - y_i)^2. \]

Kinetic energy (discrete version)

Then \( dH = \sum_i (y_i - x_i) \, dx_i - \sum_i (y_i - x_i) \, dy_i. \)

\[ \therefore \sigma(H) = \left\{ (x, y, \xi, \eta) \in T^*(\mathbb{R}^n \times \mathbb{R}^n) : \xi = \eta = y - x \right\}. \]

\( (\text{Given } (x, \xi), \text{ then } y = x + \xi, \text{ and } \eta = \xi. \)
(cont.) which is graph of \( \varphi: T^*R^n \to \),

\[ \varphi(x, \xi) = (y, \eta) = (x + \xi, \xi). \]

initial position and momentum and momentum after time 1.

\[ \therefore \text{ Linear momentum is preserved under } \varphi. \]

Newton's 1st law. (in 'quantized' time setting.)

(Discrete time is easier for doing path integral.)

A path in discrete time.

Path space = \( X^4 \).

A path in continuous time.

Path space = \( \text{Map}([0,5] \to X) \).

Trouble: Relativity \( \Rightarrow \) space is also quantized!
The world of Newton and Riemann.

This example can be generalized to Riemannian complete manifold

\[ X_1 = X_2 = (X, g). \]

Hamiltonian (discrete version)

\[ H(x, y) = -\frac{1}{2} d^2(x, y). \]

Kinetic energy (discrete version)

\[ (d(x, y) = \text{length of min. geodesic joining } x \& y) \]
ex. For \( H(x, y) = -\frac{1}{2} d^2(x, y) \),

\[
\sigma(L_H) = \text{gr}(\varphi : T^* X \to C),
\]

where \( \varphi(x) = \left( \frac{d}{dt} \left|_{t=1} \exp_x(t(\xi_x)_{x}^\#) \right) \right)^\#_g \).

the unique geodesic emanated from \( x \) in direction \((\xi_x)_{x}^\#\).

\[
\begin{align*}
T_X & \cong T^* X \\
v_{x}^g &= v \longleftrightarrow v^\#_g = v
\end{align*}
\]

use: If \( \gamma \) is a geodesic, \( \| \gamma'(s) \|_g = \text{constant} \).

\( \varphi \) gives a discrete version of geodesic flow.