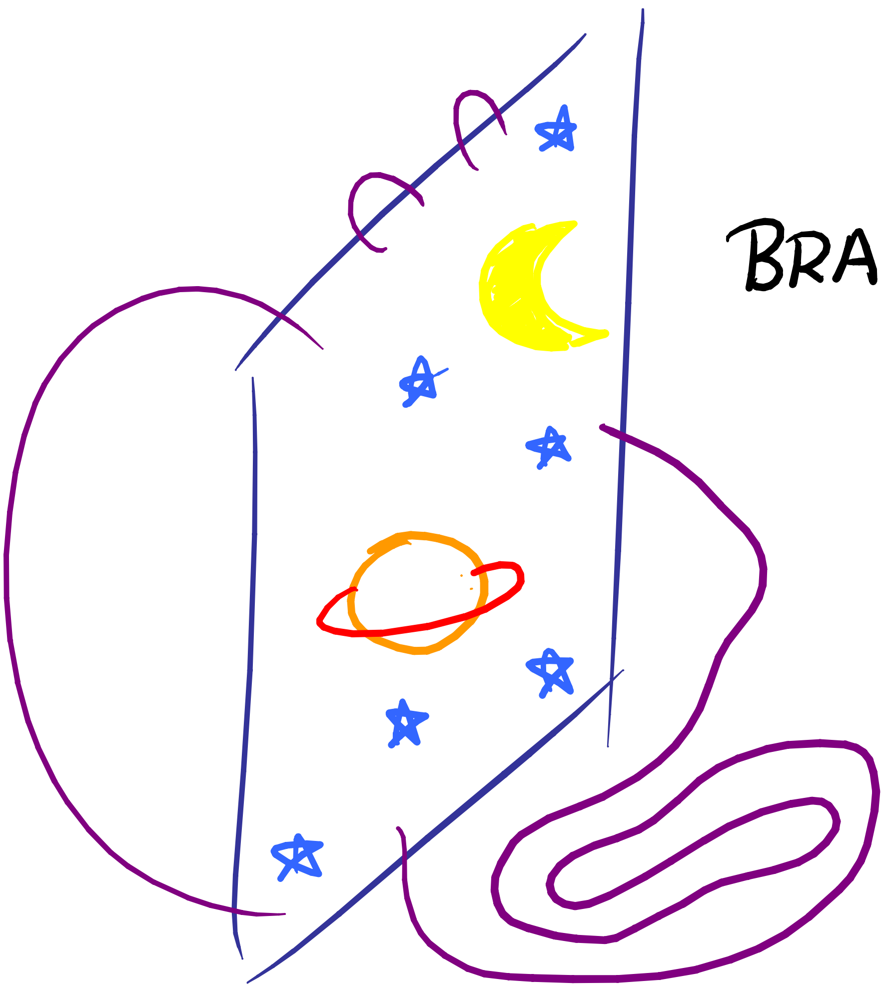
MATH 230A. Differential Geometry.

Lecture 11. Lagrangians and symplectomorphisms

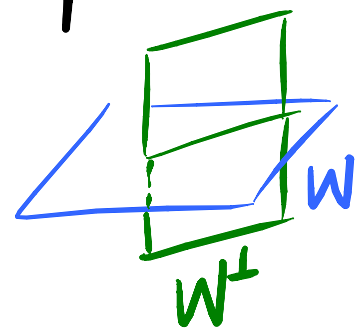
ref.: [da Silva Ch.3,4]



BRANE WORLD......

Defn: (V, ω) : symplectic vector space.

$W \underset{\text{subspace}}{\subset} V$ is



- isotropic if $W \subset W^{\perp\omega}$.
- Lagrangian if $W = W^{\perp\omega}$.
- coisotropic if $W \supset W^{\perp\omega}$.
- symplectic if $\omega|_W$ is non-degenerate.

(Recall $W^{\perp\omega} \triangleq \{v \in V : \omega(v, w) = 0 \ \forall w \in W\}$.)

ex. • $\dim W + \dim W^{\perp\omega} = \dim V$.

Consider
$$\begin{tikzcd} V \arrow[r, "\iota.\omega"] \arrow[rr, bend right] & V^* \arrow[r] & W^* \end{tikzcd}.$$

• $\mathrm{Ker} = W^{\perp\omega}$:

$$v \in \mathrm{Ker}$$
$$\Leftrightarrow (\iota_v \omega)(w) = \omega(v,w) = 0 \quad \forall w \in W$$
$$\Leftrightarrow v \in W^{\perp\omega}.$$

• $\mathrm{Im} = W^*$:

$\forall w \in W, \exists v \in V$ such that $\omega(v,w) \neq 0$.

• ker + rank $= n$.

e.x. • $(W^{\perp\omega})^{\perp\omega} = W$. ($\omega$ is non-degenerate)

• W symplectic $\Leftrightarrow W \cap W^{\perp\omega} = 0$

$\Leftrightarrow V = W \oplus W^{\perp\omega}$.

Use $\dim W + \dim W^{\perp} = V$.

• W isotropic $\Rightarrow \dim W \leq \frac{1}{2} \dim V$.

• W coisotropic $\Rightarrow \dim W \geq \frac{1}{2} \dim V$

• W Lagrangian $\Leftrightarrow W$ istropic and

$\dim W = \frac{1}{2} \dim V$.

ex. • If $W \underset{\text{isotropic}}{\subset} (V, \omega)$, then any basis of W can be extended to a symplectic basis of V.

For a basis $\{e_1, \dots, e_k\}$ of W

If $W \subsetneq W^\perp$,

pick $e_{k+1} \in W^\perp - W$.

If $\mathrm{Span}\{e_1, \dots, e_{k+1}\} \subsetneq \{e_1, \dots, e_{k+1}\}^\perp$,

pick $e_{k+2} \in \{e_1, \dots, e_{k+1}\}^\perp - \mathrm{Span}\{e_1, \dots, e_{k+1}\}$.

Inductively get $\{e_1, \dots, e_n\} = \{e_1, \dots, e_n\}^\perp$.

$\forall j$, Pick $f_j \in \underbrace{\{e_1, \dots, \hat{e}_j, \dots, e_n\}}_{n+1 \ \dim.}{}^{\perp\omega} - W$ s.t. $\omega(e_j, f_j) = 1$.

$\left(W \subsetneq \{e_1, \dots, \hat{e}_j, \dots, e_n\}^{\perp\omega}\right)$.

ex. • If $W \subset V$ is Lagrangian,

$\exists$ symplectomorphism

$$\omega_{can}(u \oplus \mu, v \oplus \nu) = \nu(u) - \mu(v).$$

$$\Phi : (V, \omega) \xrightarrow{\sim} (W \oplus W^*, \omega_{can})$$

(i.e. $\Phi^* \omega_{can} = \omega$)

with $\Phi(w) = (w, 0)$ $\forall w \in W$.

• Extend a basis $\{e_i\}$ of W to symplectic basis $\{e_i, f_i\}$.

• $\Phi(e_i) \triangleq (e_i, 0)$;

$\Phi(f_i) \triangleq (0, e_i^*)$

where $\{e_i^*\}$ is dual basis of e_i.

ex. • For $W \underset{\text{coisotropic}}{\subset} V$, $\exists$ symplectic basis $\{e_i, f_i\}$ s.t.

$$W = \operatorname{Span}\{e_1, \dots, e_n, f_1, \dots, f_k\}.$$

In particular, if $W \underset{\text{symplectic}}{\subset} V$,

$\exists$ symplectic basis $\{e_i, f_i\}_{i=1}^n$ of V such that

$$W = \operatorname{Span}\{e_1, \dots, e_k, f_1, \dots, f_k\}.$$

Easy way to remember: (follow from above)

- W isotropic $\Longleftrightarrow$ $\exists$ symplectic basis $\{e_i, f_i\}$ s.t.

  $W = \mathrm{Span}\{e_1, \dots, e_k\}$.

- W Lagrangian $\Longleftrightarrow$ $\exists$ symplectic basis $\{e_i, f_i\}$ s.t.

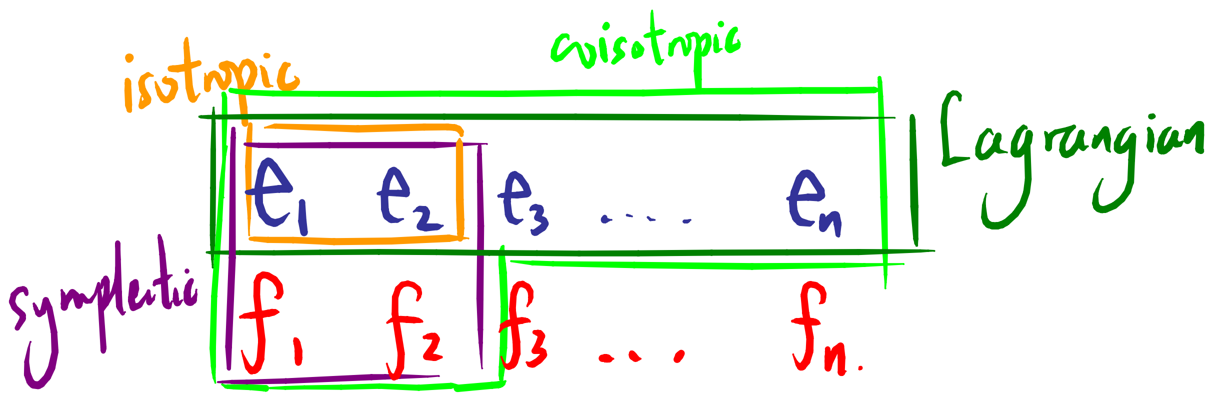
  $W = \mathrm{Span}\{e_1, \dots, e_n\}$, $n = \frac{\dim M}{2}$.

- W coisotropic $\Longleftrightarrow$ $\exists$ symplectic basis $\{e_i, f_i\}$ s.t.

  $W = \mathrm{Span}\{e_1, \dots, e_n, f_1, \dots, f_k\}$, $n = \frac{\dim M}{2}$.

- W symplectic $\Longleftrightarrow$ $\exists$ symplectic basis $\{e_i, f_i\}$ s.t.

  $W = \mathrm{Span}\{e_1, \dots, e_k, f_1, \dots, f_k\}$.



Defn.: (M,ω) symplectic.

$X \underset{\text{submfd}}{\subset} M$ is Lagrangian/isotropic/coisotropic/symplectic

if $T_xX \subset T_xM$ is.

ex. 22-8. Suppose (M,ω) is a symplectic manifold and $S \subseteq M$ is a coisotropic submanifold. An immersed submanifold $N \subseteq S$ is said to be ***characteristic*** if $T_pN = (T_pS)^\perp$ for each $p \in N$. Show that there is a foliation of S by connected characteristic submanifolds of S whose dimension is equal to the codimension of S in M.

Lagrangian submanifolds are considered as the most important class in symplectic geometry.

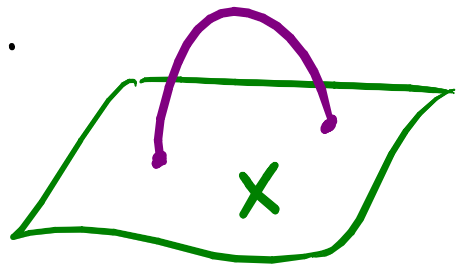
- $\underset{\text{zero-section}}{X \subset T^*X}$ is a Lagrangian submanifold.

position space   phase space

- $T_q^*X \underset{\text{fiber}}{\subset} T^*X$ is a Lagrangian submanifold.

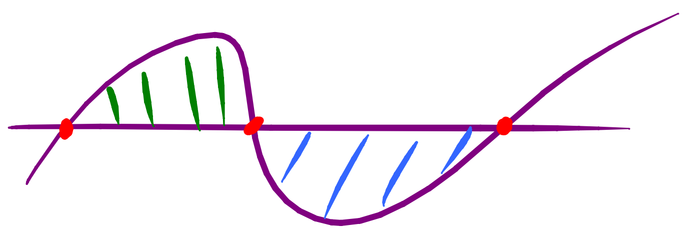
momentum space

- 'branes' in string theory.
  (∂-conditions for open string)



(Some string theorists claim that one should also consider 'coisotropic branes'.)

- Lagrangian intersection theory well-developed.

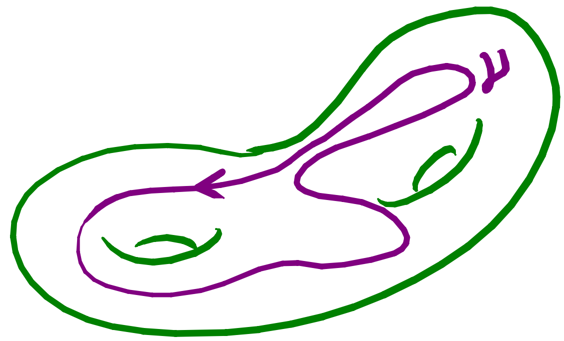


- Related to symplectomorphisms (see below).

Examples of Lagrangian submanifolds.

e.g. Any curve γ in an oriented surface is a Lagrangian submanifold.

$(\because \omega(\dot\gamma, \dot\gamma) = 0)$



e.x. (Lagrangian sections)

For (T^*X, ω_{can}) and $\varphi \in \Gamma(T^*X)$,

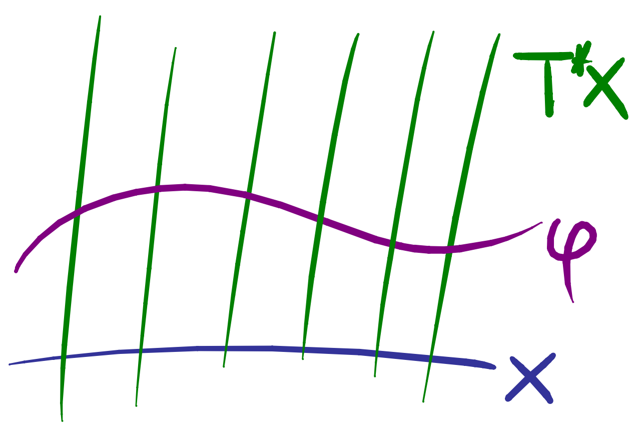
$$gr(\varphi) \underset{\text{Lag.}}{\subset} T^*X \Leftrightarrow d\varphi = 0.$$

$$\omega_{can} = \sum_{i=1}^{n} dq_i \wedge dp_i .$$

$$\varphi = \sum_{i=1}^{n} \varphi_i \, dq_i .$$

$$gr(\varphi) = \{(q, \varphi(q))\}.$$

$$\varphi^* \omega_{can} = \sum_{i=0}^{n} dq_i \wedge d\varphi_i = -d\varphi .$$



For $S \underset{\text{sub-mfd}}{\subset} X$, (conormal bundle of S)

$$N^*S \triangleq \{\nu \in T^*X : \pi(\nu) \in S,\ \nu(v) = 0\ \forall v \in TS\}.$$

$$\underset{\text{sub-bdl}}{\subset} T^*X.$$

e.g. $S = \{p\} \Rightarrow N^*S = T_p^*X$.

e.g. $S = X \Rightarrow N^*S =$ 0-section of T_p^*X.

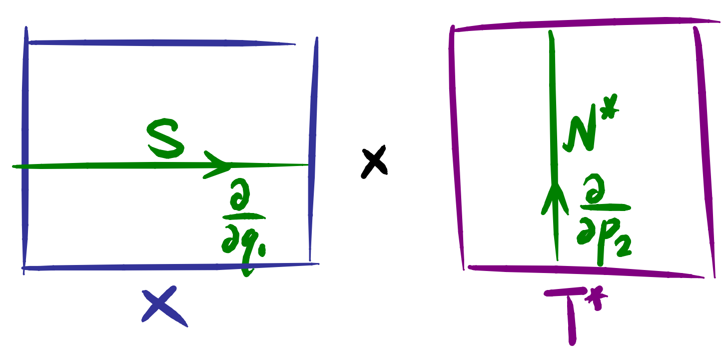
ex. $N^*S \subset T^*X$ is a Lagrangian submanifold.

Let $\{q_i\}_{i=1}^n$ be local coordinates of X s.t.

$$S = \{q_i = 0\ \forall i = k+1, \dots, n\}. \quad (k = \dim S)$$

$$N^*S = \left\{\sum_{i=1}^n p_i\, dq_i \in T^*X : p_i = 0\ \forall i = 1, \dots, k\right\}.$$

$$\omega\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_j}\right) = 0 \text{ for } i = k+1, \dots, n,\ j = 1, \dots, k.$$



local picture of T^*X

e.g. $\mathbb{C}^2, \ \omega_{std} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix}$$

ex. $T_{c_1,c_2} \triangleq \{ |z_i| = c_i \ \forall i = 1,2 \}, \ c_1, c_2 \neq 0,$

is a Lagrangian submanifold of $\mathbb{C}^2$.

For $c_1 = 0$ or $c_2 = 0$,

it is an isotropic submanifold of $\mathbb{C}^2$.

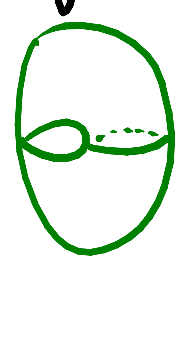
ex. $\{ |z| = |w|, \ |zw - 1| = c \}, \ c \neq 0,$

is a Lagrangian submanifold of $\mathbb{C}^2$.

($c > 0$: product tori.

$c < 0$: Chekanov tori.

$c = 0$: singular torus.)



$c = 0$.

Symplectomorphism

(M_i, ω_i) symplectic manifolds, $i = 1, 2$.

Defn. $\varphi : (M_1, \omega_1) \longrightarrow (M_2, \omega_2)$ is

symplectic if $\varphi^* \omega_2 = \omega_1$.

It is a symplectomorphism if in addition

it is a diffeomorphism.

- $\mathrm{Sympl}(M, \omega) \triangleq \{ \text{symplectomorphisms } (M, \omega) \circlearrowleft \}$

  is a subgroup of Diffeo.

Symplectic vector field

For a vector field $v \in \Gamma(TM)$,

(M compact)

It generates symplectomorphisms i.e. $(\exp tv)^* \omega = \omega$

$$\Leftrightarrow \frac{d}{dt}\left((\exp tv)^* \omega\right) = 0.$$

$$\|$$

$$(\exp tv)^* \mathcal{L}_v \omega$$

$$\Leftrightarrow \mathcal{L}_v \omega = 0.$$

<u>Defn</u> $v \in \Gamma(TM)$ is a <u>symplectic vector field</u> if $\mathcal{L}_v \omega = 0$.

e.x. $\Gamma_\omega(TM) \triangleq \{\text{Symplectic vector fields}\} \underset{\text{linear}}{\subset} \Gamma(TM)$

is closed under Lie bracket.

(Use Leibniz rule

$$\mathcal{L}_v(\mathcal{L}_w \omega) = \mathcal{L}_{\mathcal{L}_v w}\omega + \mathcal{L}_w \mathcal{L}_v \omega\ )$$

$\Gamma_\omega(TM)$ is 'Lie algebra' of $Sympl(M, \omega)$.

Hamiltonian vector fields

Def. : $v \in \Gamma(TM)$ is a Hamiltonian vector field

$$v = X_H \triangleq (dH)_{\#_\omega} \text{ for some } H \in C^\infty(M).$$

$$\left(TM \underset{(\ )_{\#_\omega}}{\overset{(\ )^{\#_\omega}}{\rightleftarrows}} T^*M \right)$$

φ is a Hamiltonian diffeomorphism if

$\varphi = \exp tv$ for some Hamiltonian vector field v, $t \in \mathbb{R}$.

(Conservation of energy)

- A Hamiltonian vector field v is symplectic.

$$\mathcal{L}_v \omega = d\iota_v \omega = ddH = 0.$$

- It preserves the Hamiltonian H:

$$\mathcal{L}_v H = 0.$$

$$\mathcal{L}_v H = dH(v) = \omega(v, v) = 0.$$

$$\{\text{Ham. v.f.}\} \underset{\#_\omega}{\longleftrightarrow} \begin{Bmatrix}\text{exact} \\ \text{1-form}\end{Bmatrix}$$

$$\cap$$

$$\{\text{Sympl. v.f.}\} \underset{\#_\omega}{\longleftrightarrow} \begin{Bmatrix}\text{closed} \\ \text{1-form}\end{Bmatrix}.$$

Poisson structure.

e.x. {Hamiltonian vector fields} is closed under Lie bracket.

$[X_f, X_g] = X_{-\{f,g\}}$, where $\{f,g\}$ is defined below.

Defn. The Poisson structure induced by ω is $\{\ ,\ \}: C^\infty(M) \otimes C^\infty(M)$ defined by

$$\{f,g\} \triangleq \omega(X_f, X_g).$$

e.x. • $\{f,g\} = X_g f = -X_f g$.

• Jacobi identity :

$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0.$$

Nöether principle

Fix (M,ω,H): Hamiltonian system. (M connected.)

Theorem: Assume $H^1(M)=0$.

$$\{\text{Conserved quantities}\}/\mathbb{R} \xleftrightarrow{\text{1-1}} \{\text{symmetries}\}$$

up to adding a constant

$f: M \longrightarrow \mathbb{R}$ s.t. $X_H \cdot f = 0$ $\qquad$ $X \in$ v.f.(M,ω) s.t. $X\cdot H = 0$

Pf: $C^\infty(M)/\mathbb{R} \xleftrightarrow{\sim}$ Ham. v.f. $\xleftrightarrow{H^1=0}$ v.f.(M,ω)

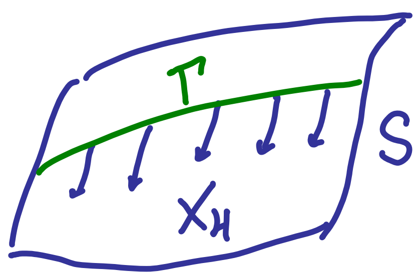
$f \longleftrightarrow X_f$

$$X_H\cdot f = 0 \Leftrightarrow \{H,f\}=0 \Leftrightarrow X_f\cdot H = 0. \quad \#$$

Hamiltonian flowout

$\Gamma \underset{\text{isotropic}}{\subset} (M, \omega)$ s.t. $\begin{cases} \Gamma \subset H^{-1}\{c\} \\ X_H \text{ nowhere tangent to } \Gamma. \end{cases}$

The flowout S from Γ along X_H is isotropic and $S \subset H^{-1}\{c\}$.



Pf: $\omega(X_H, v) = dH(v) = 0 \quad \forall v \in T\Gamma$

$\Rightarrow S$ is isotropic.

$X_H \cdot H = \{X_H, X_H\} = 0$

$\Rightarrow H|_{\text{flowlines of } X_H \text{ from } p \in \Gamma} \equiv c$

$\Rightarrow H(S) = c.$ #

Lagrangians and symplectomorphisms.

From (M_i, ω_i) symplectic manifolds, $i = 1,2$, get

$(M_1 \times M_2, \omega_\pm = \omega_1 \pm \omega_2)$ symplectic manifolds.

e.x. Let $\varphi : M_1 \longrightarrow M_2$ diffeomorphism.

φ is symplectomorphism

$\Leftrightarrow gr(\varphi) \subset (M_1 \times M_2, \omega_-)$ is Lagrangian.

$\left(\because \ \omega_- \big|_{gr(\varphi)} = \omega_1 - \varphi^* \omega_2 . \right)$

Cook up Lagrangians in $(M_1 \times M_2, \omega_-)$

Focus on $M_i = (T^*X_i, \omega_{can})$.

ex. $(M_1 \times M_2, \omega_+) \simeq (T^*(X_1 \times X_2), \omega_{can})$.

- Can cook up Lagrangian $\subset (T^*\underbrace{(X_1 \times X_2)}_{\mathbb{X}}, \omega_{can})$ by $\mathrm{graph}(\omega) \subset T^*\mathbb{X}$, where

$$\omega \in \Omega^1_{closed}(\mathbb{X}).$$

e.g. take $\omega = dH$, $H : \mathbb{X} \to \mathbb{R}$.

- How about Lagrangian $\subset (M_1 \times M_2, \omega_-)$?

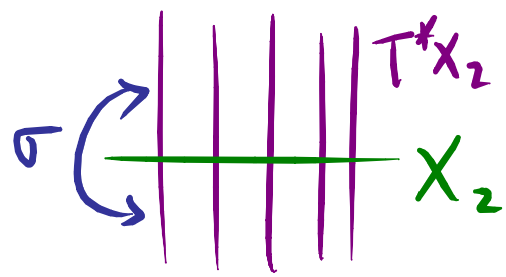
Want to have $\sigma : M_1 \times M_2 \circlearrowleft$ s.t. $\sigma^* \omega_- = \omega_+$.

Then $L \underset{Lag}{\subset} (T^*\mathbb{X}, \omega_+) \Leftrightarrow \sigma(L) \underset{Lag}{\subset} (T^*\mathbb{X}, \omega_-)$.

Involution on T^*X.

e.x. $\sigma : T^*X_2 \circlearrowleft \quad v \mapsto -v$ has the property

$$\sigma^*\omega_{can} = -\omega_{can}.$$

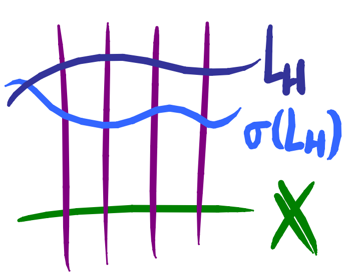


$\therefore \underbrace{(\mathrm{Id},\sigma)}_{\text{also denoted as } \sigma} : T^*X_1 \times T^*X_2 \circlearrowleft$ has the property

$$\sigma^*\omega_- = \omega_+.$$

$$\therefore L \underset{Lag}{\subset} (T^*X, \omega_+) \Leftrightarrow \sigma(L) \underset{Lag}{\subset} (T^*X, \omega_-).$$

Then $H \rightsquigarrow L_H \triangleq gr(dH) \underset{\text{Lagrangian}}{\subset} (T^*X, \omega_+)$



$$\rightsquigarrow \sigma(L_H) \underset{\text{Lagrangian}}{\subset} (T^*X, \omega_-).$$

Discrete version of Hamiltonian dynamics.

e.x. $\sigma(L_H) \underset{\text{Lag.}}{\subset} (T^*X, \omega_-) = gr(\varphi : T^*X_1 \xrightarrow{\sim} T^*X_2)$

(global condition)

$\Rightarrow \partial^{X_1}\partial^{X_2} H\big|_{(x_1,x_2)} : T_{x_1}X_1 \otimes T_{x_2}X_2 \longrightarrow \mathbb{R}$

(local condition) non-degenerate $\forall (x_1, x_2) \in X_1 \times X_2$.

$\forall\ (\xi_x, \eta_y) \in T^*X_1 \times T^*X_2$,

given ξ_x, solve for y.

$$\eta_y = \varphi(\xi_x) \iff \begin{cases} \xi_x = d_x H(x,y) \\ \eta_y = -d_y H(x,y) \end{cases}$$

then get η_y

and hence show that

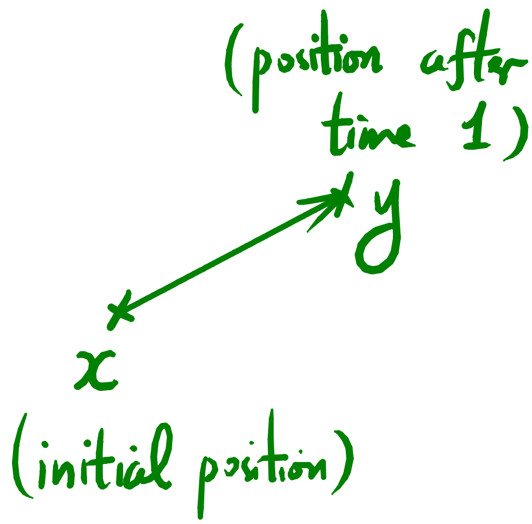
$\det(\partial_{x_i}\partial_{y_j} H) \neq 0$ by taking derivative

on RHS and use φ is a diffeomorphism.

e.g. $X_1 = X_2 = \mathbb{R}^n$.

Phase space (discrete version):

$\mathbb{R}^n \times \mathbb{R}^n$.



Hamiltonian (discrete version):

$$H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},$$

$$H(x,y) = -\frac{1}{2}|x-y|^2.$$

Kinetic energy (discrete version)

$$= \sum_i \frac{-1}{2}(x_i - y_i)^2.$$

Then $dH = \sum_i (y_i - x_i)\, dx_i - \sum_i (y_i - x_i)\, dy_i$.

$$\therefore \sigma(L_H) = \{ (x, y, \xi, \eta) \in T^*(\mathbb{R}^n \times \mathbb{R}^n) : \xi = \eta = y - x \}.$$

(Given (x, ξ), then $y = x + \xi$, and $\eta = \xi$.)

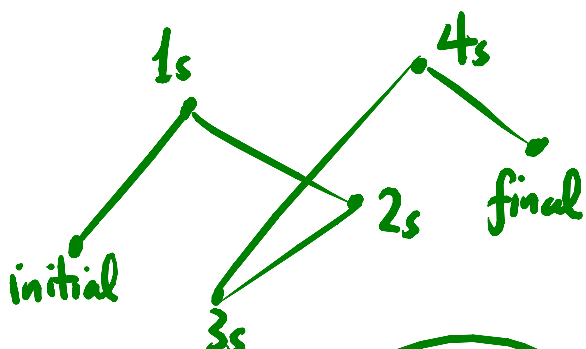
(cont.) which is graph of $\varphi : T^*\mathbb{R}^n \circlearrowleft$,

$$\varphi(x,\xi) = \underbrace{(y,\eta)}_{\substack{\text{initial position} \\ \text{and momentum}}} = \underbrace{(x+\xi,\xi)}_{\substack{\text{position and momentum} \\ \text{after time 1.}}}$$

$\therefore$ Linear momentum is preserved under φ.

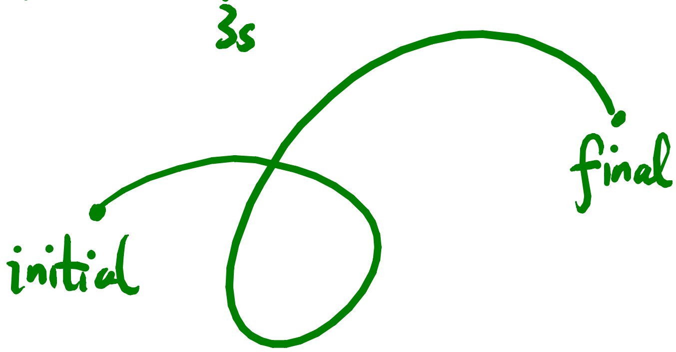
Newton's 1^{st} law. (in 'quantized' time setting.)

(Discrete time is easier for doing path integral.)



A path in discrete time.

Path space $= X^4$.



A path in continuous time.

Path space $= \mathrm{Map}([0,5] \to X)$.

Trouble: Relativity $\Rightarrow$ space is also quantized!

The world of Newton and Riemann.

This example can be generalized to

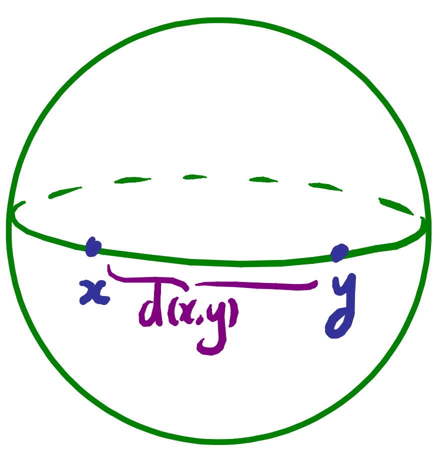
Riemannian complete manifold

$$X_1 = X_2 = (X, g).$$

Hamiltonian (discrete version) $\quad H(x,y) = -\frac{1}{2} d^2(x,y).$

Kinetic energy (discrete version)

$\left(d(x,y) \triangleq \text{length of min. geodesic joining } x \ \& \ y \right)$

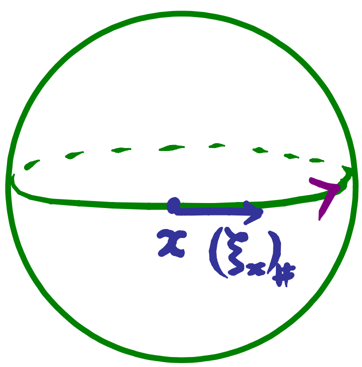


ex. For $H(x,y) = -\frac{1}{2} d^2(x,y)$,

$$\sigma(L_H) = gr\left(\varphi : T^*X \circlearrowleft\right),$$

where

$$\varphi(\xi_x) = \left(\frac{d}{dt}\Big|_{t=1} \underbrace{\exp_x\left(t(\xi_x)_{\#_g}\right)}\right)^{\#_g}.$$



the unique geodesic emanated from x in direction $(\xi_x)_\#$.

$$TX \underset{g}{\cong} T^*X$$

$$\nu_{\#_g} = v \in TX \longleftrightarrow v^{\#_g} = \nu \in T^*X$$

use: If γ is a geodesic, $\|\gamma'(s)\|_g = $ constant.

φ gives a discrete version of geodesic flow.