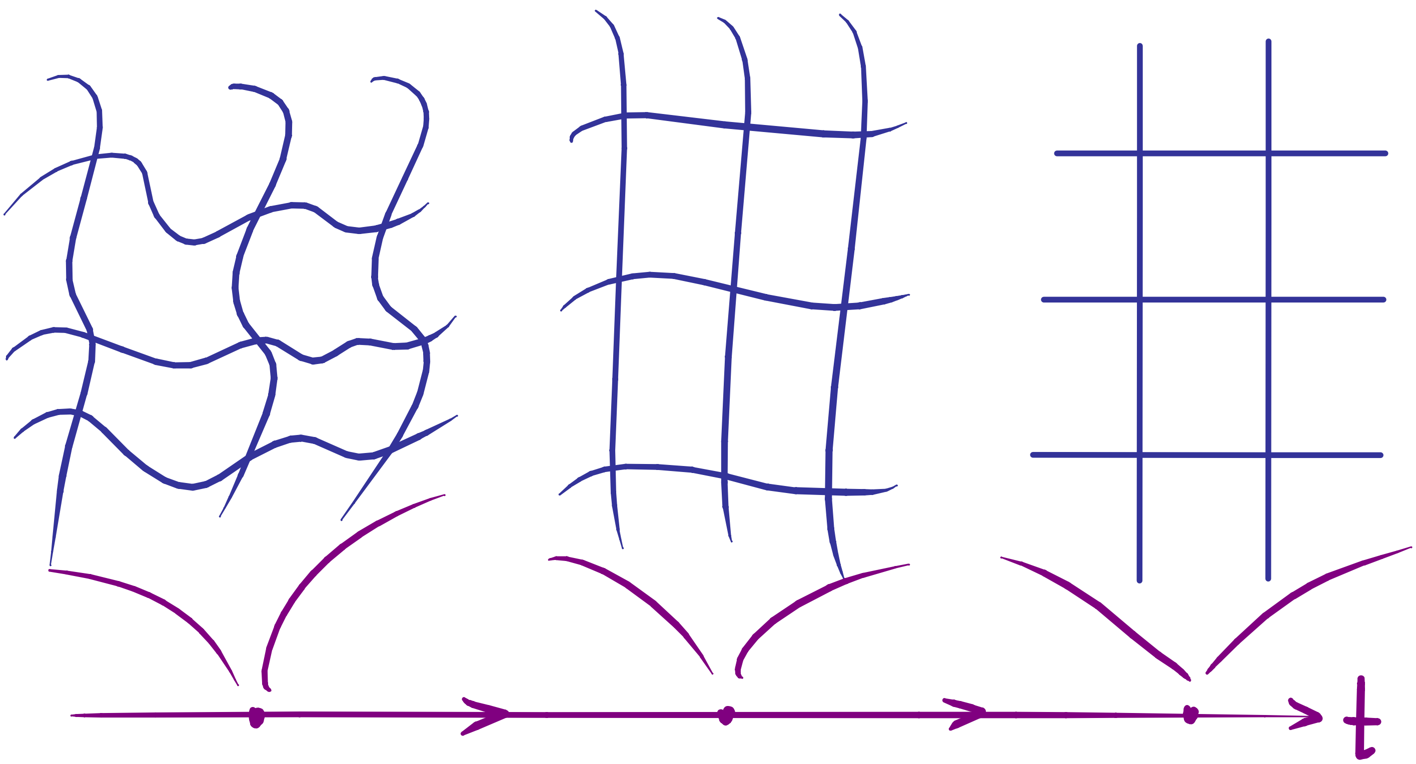
MATH 230A. Differential Geometry.

Lecture 13. Moser argument.

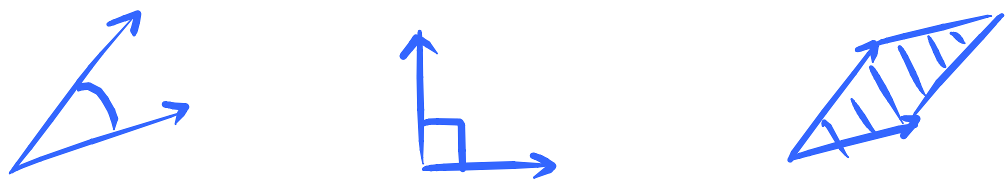
ref.: [da Silva Ch. 7]



Motivation

Standardization of geometric structures:

(or 'adapted coordinates')



	metric g	almost complex structure J	symplectic structure ω
Local frame of tangent bundle trivializing the geometric structures	$\{e_i\}$ s.t. $g(e_i,e_j)=\delta_{ij}$.	$\{e_i,f_i\}$ s.t. $J\cdot e_i=f_i$; $J\cdot f_i=-e_i$	$\{e_i,f_i\}$ s.t. $\omega=\sum e_i^*\wedge f_i^*$.
Local coordinate frame $\{\frac{\partial}{\partial x_i}\}$ trivializing the geometric structures	If exist, g is flat metric. Obstruction: curvature.	If exist, J is a complex structure. Obstruction: Nijenhuis tensor.	Always exist!

To compare two symplectic structures (M_i, ω_i), $i=0,1$:

(1.) Symplectomorphic: $\exists\, \varphi : M_0 \xrightarrow[\simeq]{\text{diffeo.}} M_1$

s.t. $\varphi^* \omega_1 = \omega_0$.

Otherwise try to deform one symplectic structure to another:

(2.) Deformation equivalent: $\exists$ symplectic forms ω_t, $t \in [0,1]$.

(smooth on t)

(3.) Isotopic: $\exists$ symplectic forms ω_t, $t \in [0,1]$.

s.t. $[\omega_t] = [\omega_0] \in H^2(M, \mathbb{R})$ $\forall t$.

- Isotopy is relatively easy to construct.

e.g. consider $\omega_t \triangleq \omega_0 + t(\omega_1 - \omega_0)$ (hope it non-degenerate).

Moser: It turns out such isotopy $\Rightarrow$ $\omega_0 \underset{\text{symplectomorphic}}{\simeq} \omega_1$

$\therefore$ $[\omega]$ 'almost' determine ω!

Moser constructs the symplectomorphism by considering:

(4.) Strongly isotopic: $\exists\, \rho_t : M_0 \xrightarrow[\sim]{\text{diffeo.}} M_1$, $t \in [0,1]$,

s.t. $\rho_0 = \mathrm{Id}$; $\rho_1^* \omega_1 = \omega_0$.

ex. $(4) \Rightarrow (3) \Rightarrow (2)$.
$\hookrightarrow (1)$.

Non-trivial part is $(4) \Rightarrow (3)$. Take $\omega_t = \rho_t^* \omega_0$.

$\rho_t^* \omega_0 - \omega_0 = \int_0^t \left(\frac{d}{ds} \rho_s^* \omega_0 \right) ds$, and use Cartan formula.

Moser goes from (2) to (4).

Moser theorem: ω_0, ω_1: symplectic form on M. (compact)

Suppose $[\omega_0] = [\omega_1]$ and $\forall t \in [0,1]$,

$$\omega_t \triangleq (1-t)\,\omega_0 + t\,\omega_1 \text{ is non-degenerate.}$$

Then $\exists$ strong isotopy $\rho_t : M \underset{\text{diffeo.}}{\circlearrowleft \simeq}$ s.t.

$$\rho_t^* \omega_t = \omega_0 .$$

Pf.: Solve for ρ_t from the equation $\rho_t^* \omega_t = \omega_0$.

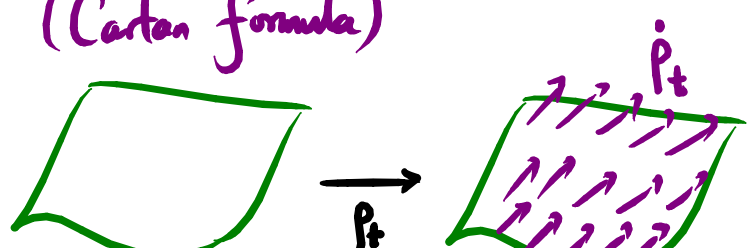
ρ_t is determined by ρ_0 and the vector fields $\dot\rho_t$.

(1.) $\frac{d}{dt}$ on $\rho^* \omega_t = \omega_0$:

$$\underbrace{\left(\frac{d}{dt}\rho_t^*\right)}_{\cancel{(\rho_t^* \circ \iota_{\dot\rho_t}) \circ d} + d \circ (\rho_t^* \circ \iota_{\dot\rho_t})} \omega_t + \rho_t^* \overbrace{\frac{d}{dt}\omega_t}^{\omega_1 - \omega_0} = 0 .$$

(Cartan formula)

$d\omega_t = 0$



(cont.) (2.) Want $d\left(\iota_{\dot{\rho}_t}\omega_t\right) = -(\omega_1 - \omega_0)$.

Since $[\omega_1 - \omega_0] = 0 \in H^2(M)$,

$\exists \mu \in \Omega^1(M)$ s.t. $d\mu = -(\omega_1 - \omega_0)$.

Take $\iota_{\dot{\rho}_t}\omega_t = \mu$.

(3.) Since ω_t is non-degenerate,

$\exists v_t \in \Gamma(T)$ s.t. $\iota_{v_t}\omega_t = \mu$.

Then take ρ_t to be the isotopy generated by v_t. #

ex. Show that the above theorem still holds for general symplectic forms ω_t, $t \in [0,1]$, with

$$\left[\frac{d}{dt}\omega_t\right] = 0.$$

Relative Moser theorem.

Given $X \subset M$ and ω_0, ω_1 on M which agrees on X, then they are symplectomorphic in a tubular neighborhood of X.

Tubular neighborhood theorem:

Let $X \underset{\text{submfd}}{\subset} M$.

$\exists$ convex neighborhood $U_0 \underset{\text{open}}{\subseteq} NX$ of X

and $\varphi : U_0 \hookrightarrow M$, $\underbrace{\varphi(U_0)}_{\text{tubular neighborhood of } X} \underset{\text{open}}{\subseteq} M$.

$$\begin{tikzcd} U_0 \arrow[rr, hook] & \circlearrowright & M \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

To do Moser argument, need a family vesion of Poincaré Lemma.

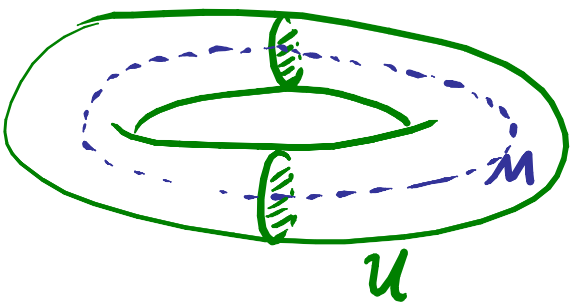
Prop: (Family version of Poincaré lemma)

Let $X \underset{\text{submfd}}{\subset} M$, and $U \underset{\text{open}}{\subset} M$ be a tubular neighborhood of X.

Notice that this is stronger than $\iota_X^* \nu = 0$.

Suppose $\nu \in \Omega^k_{\text{closed}}(U)$ with $\nu|_X = 0$. Then

$$\nu = d\eta \text{ for some } \eta \in \Omega^{k-1}(U) \text{ with } \eta|_X = 0.$$



U is a tubular neighborhood of X

$\Leftrightarrow \exists\, U_0 \underset{\text{convex}}{\subset} NX \triangleq \dfrac{TM|_X}{TX}$ s.t.

one have the diagram
$$\begin{tikzcd} U \arrow[r, "\simeq"] & U_0 \\ X \arrow[u, hook] \arrow[ur] \arrow[u, phantom, shift left=4ex, "\circlearrowleft"] & \end{tikzcd}.$$

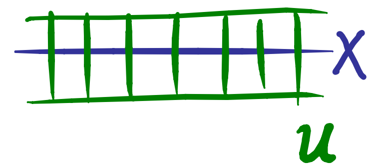
Pf.: Let $\rho_t : U \circlearrowleft (x,v) \longmapsto (x, tv)$ for $t \in [0,1]$.

(well-defined since U_0 is convex)

W.L.O.G. can work on $U \supset X$.

$\pi : U \longrightarrow X$ bundle map

$\iota : X \longrightarrow U$ inclusion.



$$(\iota \circ \pi)^* - \mathrm{Id}_U^* = dQ + Qd$$

where $Q(\nu) \triangleq \int_0^1 \iota_{\frac{d}{dt}} \rho^* \nu \, dt$

$U \times [0,1] \xrightarrow{\rho} U$

$\rho((x,v),t) = (x, tv).$

Then $\overbrace{\cancel{(\iota \circ \pi)^* \nu}}^{0 \text{ since } \nu|_X = 0} - \nu = d\,Q(\nu)$

$$\therefore \ \nu = d\eta$$

where $\eta \triangleq -Q(\nu)$. Since $\nu|_X = 0$,

$$\eta|_X = -\int_0^1 \iota_{\frac{d}{dt}} \rho^* \nu|_X \, dt = 0.$$

e.x. (Relative Moser theorem)

$X \underset{\substack{\text{compact} \\ \text{manifold}}}{\subset} M$. ω_0, ω_1 symplectic form on M.

Suppose $\omega_0(x) = \omega_1(x)$ $\forall x \in X$.

Then $\exists$ tubular neighborhood U_0, U_1 of X and

$\rho : U_0 \xrightarrow[\text{diffeo.}]{\cong} U_1$ with $\rho|_X = Id_X$

s.t. $\rho^* \omega_1 = \omega_0$.

$\omega_t \triangleq \omega_0 + t(\omega_1 - \omega_0)$ are symplectic for some $U \supset X$.

Then try to solve for $\rho_t^* \omega_t = \omega_0$ and use family version of Poincaré lemma.

e.x. (Darboux theorem)

$\forall p \in (M, \omega)$, $\exists$ coordinate chart $(\mathcal{U}, (x_i, y_i))$ such that

$$\omega|_{\mathcal{U}} = \sum_i dx_i \wedge dy_i.$$

Use any chart at p first.

Standardize ω at p by linear algebra.

Then compare ω with $\omega_0 = \sum_i dx_i \wedge dy_i$ and use relative Moser theorem.