MATH 230A. Differential Geometry.


ref. : [da Silva Ch. 7]
Motivation

Standardization of geometric structures: (or ‘adapted coordinates’)

<table>
<thead>
<tr>
<th>Metric $g$</th>
<th>Almost complex structure $J$</th>
<th>Symplectic structure $\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${e_i}$ s.t. $g(e_i, e_j) = \delta_{ij}$</td>
<td>${e_i, f_i}$ s.t. $J e_i = f_i$, $J f_i = -e_i$</td>
<td>${e_i, f_i}$ s.t. $\omega = \sum e_i^* \wedge f_i^*$</td>
</tr>
</tbody>
</table>

Local frame of tangent bundle trivializing the geometric structures

Local coordinate frame $\{\alpha\}$ trivializing the geometric structure

If exist, $g$ is flat metric
Obstruction: curvature.

If exist, $J$ is a complex structure
Obstruction: Nijenhuis tensor.

Always exist!
To compare two symplectic structures \((M_i, \omega_i)\), \(i=0,1\):

1. **Symplectomorphic**: \(\exists \, \rho : M_0 \xrightarrow{\text{diffeo.}} M_1\)
   \[\rho^* \omega_1 = \omega_0.\]

   Otherwise try to deform one symplectic structure to another:

2. **Deformation equivalent**: \(\exists\) symplectic forms \(\omega_t, t \in [0,1].\)
   \(\text{(smooth on} \ t)\)

3. **Isotopic**: \(\exists\) symplectic forms \(\omega_t, t \in [0,1].\)
   \[\text{s.t.} \, [\omega_t] = [\omega_0] \in H^2(M, \mathbb{R}) \, \forall t.\]

   - **Isotopy is relatively easy to construct.**

   e.g. consider \(\omega_t \equiv \omega_0 + t(\omega_2 - \omega_0)\) (hope it non-degenerate).

**Moser**: It turns out such isotopy \(\Rightarrow \omega_0 \sim \omega_1\), symplectomorphic.

\[\therefore [\omega] \text{ ‘almost’ determine } \omega!\]
Moser constructs the symplectomorphism by considering:

(4.) **Strongly isotopic:** \( \exists \varphi_t : M_0 \xrightarrow{\text{diffeo.}} M_1, t \in [0,1], \)

\[ \varphi_0 = \text{Id}; \varphi_t^* \omega_1 = \omega_0. \]

**ex.** (4) \( \Rightarrow \) (3) \( \Rightarrow \) (2).

\[ \Leftrightarrow \] (1).

Non-trivial part is (4) \( \Rightarrow \) (3). Take \( \omega_t = \varphi_t^* \omega_0. \)

\[ \varphi_t^* \omega_0 - \omega_0 = \int_0^t \left( \frac{d}{ds} \varphi_s^* \omega_0 \right) ds, \] and use Cartan formula.

Moser goes from (2) to (4).
Moser theorem: $\omega_0, \omega_1$: symplectic form on $M$. (compact) Suppose $[\omega_0] = [\omega_1]$ and $\forall t \in [0, 1]$, $\omega_t = (1-t)\omega_0 + t\omega_1$ is non-degenerate. Then $\exists$ strong isotopy $p_t: M \cong$ s.t. $p_t^* \omega_t = \omega_0$.

Pf.: Solve for $p_t$ from the equation $p_t^* \omega_t = \omega_0$.

$p_t$ is determined by $p_0$ and the vector fields $\dot{p}_t$.

\[ (1) \frac{d}{dt} \text{ on } p_t^* \omega_t = \omega_0: \]

\[ (\frac{d}{dt} p_t^*) \omega_t + p_t^* \frac{d}{dt} \omega_t = 0. \]

\[ \text{(Cartan formula)} \]

\[ (p_t^* \circ \dot{p}_t) \circ d + d (p_t^* \circ \dot{p}_t) \]

$\text{d} \omega_t = 0$
(2.) Want $d(\gamma_{Pt} \omega_t) = -(\omega_t - \omega_0).

Since $[(\omega_t - \omega_0)] = 0 \in H^2(M),
\exists \mu \in \Omega^4(M) \text{ s.t. } d\mu = -(\omega_t - \omega_0).
Take $\gamma_{Pt} \omega_t = \mu.

(3.) Since $\omega_t$ is non-degenerate,
\exists \nu_t \in \Gamma(T) \text{ s.t. } \gamma_{\nu_t} \omega_t = \mu.

Then take $p_t$ to be the isotopy generated by $\nu_t$.

Ex. Show that the above theorem still holds for general symplectic forms $\omega_t$, $t \in [0,1]$, with

$$\left[ \frac{d}{dt} \omega_t \right] = 0.$$
Relative Moser theorem.

Given $X \subset M$ and $\omega_0, \omega_1$ on $M$ which agrees on $X$, then they are symplectomorphic in a tubular neighborhood of $X$.

Tubular neighborhood theorem:

Let $X \subset M$.

There exists a convex neighborhood $U_0 \subset \text{open } NX$ of $X$.

and $\psi: U_0 \to M$, $\psi(U_0) \subset \text{open } M$.

To do Moser argument, need a family version of Poincaré Lemma.
Prop: (Family version of Poincaré lemma)

Let $X \subseteq M$, and $U \subseteq M$ be a tubular neighborhood of $X$.

Suppose $\omega \in \Omega^k_{\text{closed}}(U)$ with $\omega|_X = 0$. Then

$\omega = d\eta$ for some $\eta \in \Omega^{k-1}(U)$ with $\eta|_X = 0$. 

(U is a tubular neighborhood of $X$)

$\iff \exists U_0 \subseteq N_X \cong \frac{TM|_X}{TX}$ st.

one have the diagram

\[
\begin{array}{c}
U \\
\downarrow \phi \\
X
\end{array}
\]

Pf.: Let $p_t: U \supset (x,v) \mapsto (x,tv)$ for $t \in [a,1]$.

(well-defined since $U_0$ is convex)

W.L.O.G. can work on $U \supset X$. 

Notice that this is stronger than $\iota_* \iota^* \omega = 0$. 

\( \pi : U \longrightarrow X \) bundle map

\( \iota : X \longrightarrow U \) inclusion.

\((\iota \circ \pi)^* - \text{Id}_U = dQ + Qd\)

where \( Q(v) = \int_0^1 \frac{d}{dt} \rho^t v \, dt \)

\( U \times [0,1] \xrightarrow{p} U \)
\( p((x,v),t) = (x, tv) \).

Then  \((\iota \circ \pi)^* \nu - \nu = dQ(v)\)

\( \therefore \nu = d\eta \)

where \( \eta = -Q(v) \). Since \( \nu|_x = 0 \),

\( \eta|_x = -\int_0^1 \frac{d}{dt} \rho^t v|_x \, dt = 0. \)
\textbf{ex. (Relative Moser theorem)}

Let \( X \subseteq M \) be a compact manifold with \( \omega_0, \omega_1 \) symplectic forms on \( M \).

Suppose \( \omega_0(x) = \omega_1(x) \) \( \forall x \in X \).

Then there exists a tubular neighborhood \( U_0, U_1 \) of \( X \) and a diffeomorphism \( p : U_0 \to U_1 \) with \( p|_x = \text{Id}_x \)

such that \( p^* \omega_1 = \omega_0 \).

\( \omega_t = \omega_0 + t(\omega_1 - \omega_0) \) are symplectic for some \( t \geq 0 \).

Then try to solve for \( p_t^* \omega_t = \omega_0 \) and use the family version of Poincaré lemma.
(Darboux theorem)

\[ \forall p \in (M, \omega), \exists \text{ coordinate chart } (U, (x_i, y_i)) \text{ such that } \quad \omega|_U = \sum_i dx_i \wedge dy_i. \]

Use any chart at \( p \) first.

Standardize \( \omega \) at \( p \) by linear algebra.

Then compare \( \omega \) with \( \omega_0 = \sum_i dx_i \wedge dy_i \) and use relative Moser theorem.