MATH 230A. Differential Geometry

Lecture 14. Weinstein neighborhood theorem

ref.: [da Silva Ch. 5]
Weinstein Lagrangian neighborhood theorem.

\( \omega_1, \omega_2 \) symplectic on \( M \).

\( X \subset M \) Lagrangian for both \( \omega_1, \omega_2 \).

Then \( \exists \) tubular neighborhood \( U_1, U_2 \) of \( X \) and \( \varphi : (U_1, \omega_1) \to (U_2, \omega_2) \) (means \( \varphi^* \omega_2 = \omega_1 \)) with \( \varphi|_X = \text{Id} \).
Want to use relative Moser theorem:

\[ X \subset M, \ w_0, w_1 \text{ symplectic form on } M. \]

Suppose \( w_0(x) = w_1(x) \ \forall x \in X. \)

Then \( \exists \) tubular neighborhood \( U_0, U_1 \) of \( X \) and \( \rho : U_0 \xrightarrow{\text{diffeo.}} U_1 \) with \( \rho|_x = \text{Id}_x \)

st. \( \rho^* w_1 = w_0. \)

Known: \( l_x^* w_1 = l_x^* w_2 = 0. \)

Need: \( w_1|_x = w_2|_x. \)
Strategy

① Show \( \exists \varphi_x : T_x M \) linear such that smoothly depending on \( x \in X \)

\[
\begin{align*}
\{ \varphi_x \big|_{T_x X} &= \text{Id}, \\
(\varphi_x^* \omega_2)(x) &= \omega_1(x).
\}
\]

② \( \varphi_x \) integrates to a tubular neighborhood \( N \) of \( x \) to be \( \varphi : N \to M \) such that

\[
\varphi \big|_x = \text{Id} \quad \text{and} \quad (\varphi^* \omega_2)^x = (\omega_1)^x.
\]
Lemma 1: \((V, \omega)\) symplectic vector space. 
\[ L \leq V. \]

Given \(W < V\) with 
\[ W \oplus L = V, \]
Can construct \(W' \leq V\) with \(W' \cap L\).
(‘modify’ \(W\) to make it Lagrangian)

Pf.: \(\cdot\) Want to take 
\[ W' = \{ w + Aw : w \in W \} \]
for some \(A: W \to L\). 
(‘tilt’ \(W\) by \(L\).)
(cont.) \[ W \quad \text{Lag.} \]

\[ \Rightarrow \quad w(w_{1} + Aw_{1}, w_{2} + Aw_{2}) = 0 \quad \forall w_{1}, w_{2} \in W. \]

\[ \begin{align*}
&\Rightarrow w(w_{1}, w_{2}) + w(w_{1}, Aw_{2}) + w(Aw_{1}, w_{2}) \\
&( \text{L Lagrangian } \Rightarrow w(Aw_{1}, Aw_{2}) = 0 )
\end{align*} \]

\[ \begin{align*}
&\Rightarrow w(w_{1}, w_{2}) = -w(w_{1}, w_{2}). \\
&\Leftrightarrow \quad w_{1}(w_{1}, w_{2}) = -w_{1}(w_{1}, w_{2}).
\end{align*} \]

\[ \therefore \text{Want to define } A : W \quad \text{linear} \quad \rightarrow L \text{ such that} \]

\[ \begin{align*}
&\Rightarrow w_{1}(w_{1}, Aw_{2}) + w_{1}(Aw_{1}, w_{2}) = -w_{1}(w_{1}, w_{2}). \\
&\Leftrightarrow \quad w_{1}(w_{1}, Aw_{2}) = -(w_{1}(w_{1}, w_{2})).
\end{align*} \]

\[ \cdot \quad \text{w gives } L \quad \rightarrow W^{*}. \quad \text{w} \quad \rightarrow \quad W^{*} \]

\[ \Leftrightarrow \quad \text{Same dimension} \]

\[ \cdot \quad \text{injective : } w_{1}w_{1|w} = 0 \Rightarrow w_{1} \in W^{1, w} \]

\[ \text{But } w_{1} \in L = L^{1, w} \]

\[ \therefore w_{1} \in (L + W)^{1, w} = V^{1, w} \Rightarrow v = 0. \]
(cont.) \[ L \overset{\sim}{\rightarrow} W^*. \]

\[ \forall w \in W, \quad \exists v \in L \text{ s.t. } \omega \omega = -\frac{1}{2} \omega(w, \cdot) \epsilon W^*. \]

Define \( A \cdot w \buildrel \triangle \over = v \). (obviously linear.)

Then \( \forall w_1, w_2 \in W, \)

\[ \omega(w_1, Aw_2) + \omega(Aw_1, w_2) = -\omega(w_1, w_2). \]

\( \therefore W' \overset{\triangle}{=} \{ w + Aw : w \in W \} \) is Lagrangian.
ex $w_1, w_2$ symplectic on $V$, and $L \subset V$ Lagrangian with respect to both $w_1, w_2$. Then every $W \subset V$ with $W \oplus L = V$ gives $\varphi : (V, w_1) \overset{\sim}{\rightarrow} (V, w_2)$ with $\varphi |_L = \text{Id}$.

- Lemma 1 $\Rightarrow W_i \subset V$ with $W_i \cap L$, $i=1, 2$.

- Have $W_1 \overset{\varphi}{\rightarrow} L^* \overset{\varphi}{\leftarrow} W_2$.

Take $\varphi = \text{Id} + \uparrow : V = L \oplus W_1 \rightarrow L \oplus W_2 = V$. 
ex. \( \omega_1, \omega_2 \) symplectic on \( M \).

\( \mathcal{X} \subset M \) Lagrangian for both \( \omega_1, \omega_2 \).

Then \( \exists \varphi_x : T_xM \rightarrow \text{linear} \) such that

\[
\begin{align*}
\varphi_x \big|_{T_xX} &= \text{Id}. \\
\varphi_x^* \omega_2(x) &= \omega_1(x).
\end{align*}
\]

Use metric to choose a sub-bundle \( W \subset TML_x \) such that \( W + TX = TML_x \).

Then use the previous exercise.
Whitney extension theorem

$X \subset M$. 

Let $\eta : TM|_X \sim \to TM|_X$ with $\eta|_{T_x} = \text{Id}$. (bundle isomorphism)

Then $\exists$ tubular neighborhood $N \ni X$ and $\nabla \xrightarrow{h} M$ such that $h^*|_X = \eta : TM|_X \subset$.

(Use metric and exponentiate $\eta$ to get $h$.)

e.x. Use Whitney extension theorem to prove Weinstein neighborhood theorem.

(See Step 2.)
For $X \subset (M, \omega)$, Lagrangian

\exists \text{ neighborhood } U_0 \text{ of } X \subset T^*X \text{ and } y : (U_0, \omega_{can}) \rightarrow (M, \omega).

\[ \text{\begin{align*} & Y X \cong T^*X. \\ & \text{Use tubular neighborhood theorem and} \\ & \text{Weinstein neighborhood theorem.} \end{align*}} \]
Application to symplectomorphisms.

\[ \Phi : (M, \omega) \to \cdot \]

\[ \implies \text{graph}(\Phi) \subseteq (M \times M, \omega_\cdot) \text{ Lagrangian} \]

Also diagonal \[ \Delta \subseteq (M \times M, \omega_\cdot) \text{ Lagrangian} \]

Weinstein tubular neighborhood theorem

\[ \implies \text{Tubular neighborhood } U_0 \subseteq T^* \Delta \text{ and } \]

\[ \Phi : (U_0, w_{can}) \to (M \times M, \omega_\cdot) \text{ Lagrangian} \]
Now if \( \Phi \sim \text{Id} \) sufficiently, \( \text{graph}(\Phi) \subset \varphi(U) \) and 
\[
\varphi^{-1}(\text{graph}(\Phi)) = \text{graph}(\eta)
\]

\[
\text{closed 1-form on } \Delta = M
\]

\[
\begin{array}{c}
\text{Lie algebra of } \text{Symp}(M,\omega) \text{ is } \Omega^1_{\text{closed}}(M).
\end{array}
\]

\[
\text{e.g. } \{\text{Symplectic vector fields} \} \sim 2\omega \Omega^1_{\text{closed}}(M).
\]
ex. \{Fixed points of \overline{\Phi}\} = \{zeros of \eta\}.

Suppose \(H^4(M) = 0\).

Then \(\eta = df\).

If \(M\) is compact, \(f\) has at least two critical points (maximum/minimum points). 

\(\therefore\ \overline{\Phi}\) has at least two fixed points!
Arnold conjecture: (proved)

If \( \varphi = \varphi_t \) for \( \varphi_t \): isotopy generated by \( X_{H_t} \), where \( H_t \) is time-dependent with \( H_t = H_{t+1} \), then

\[
\text{# of fixed points of } \varphi \geq \sum_{i=0}^{2n} h^i(M, R).
\]