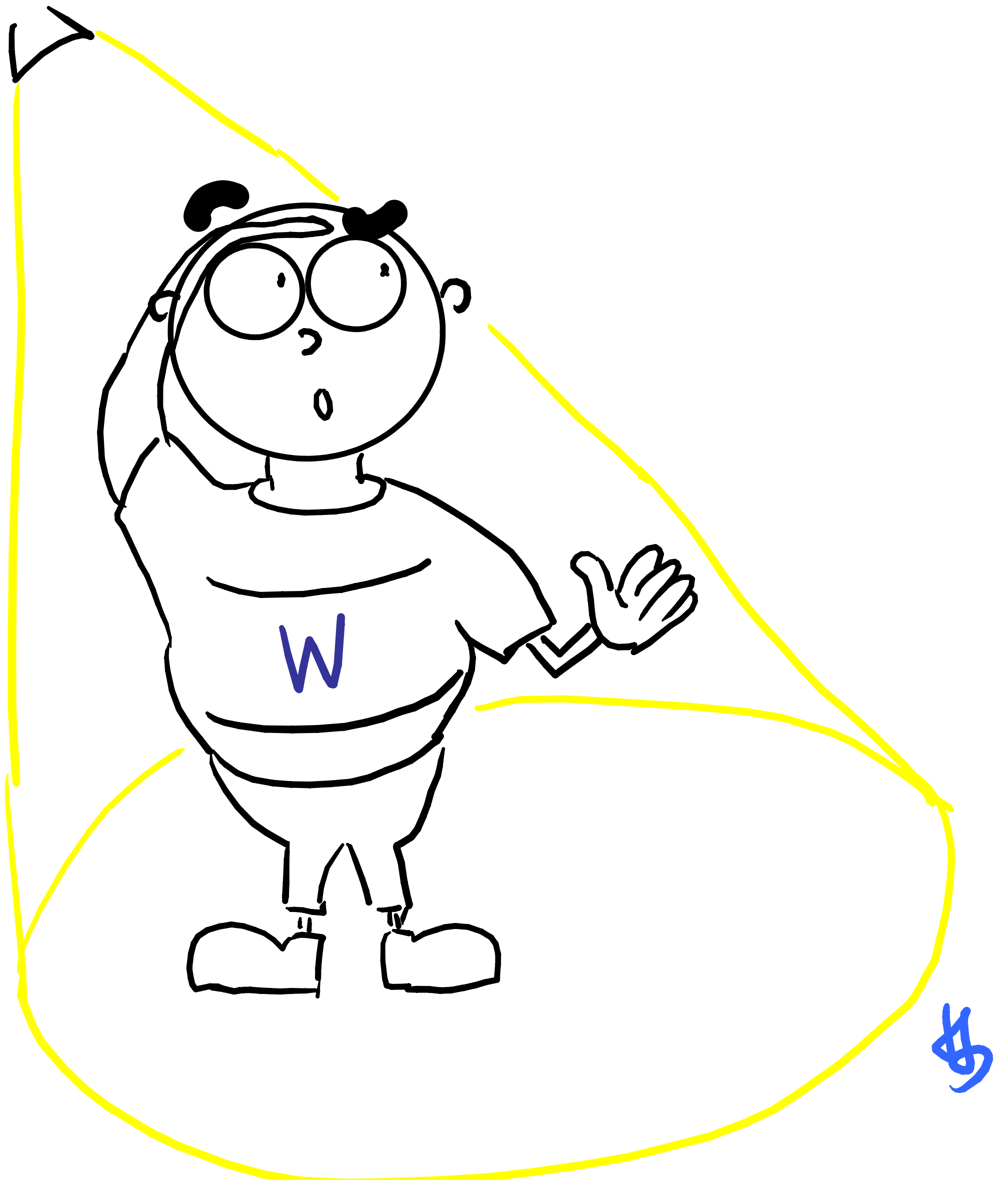
MATH 230A. Differential Geometry.

Lecture 14. Weinstein neighborhood theorem

ref. : [da Silva Ch. 5]



Weinstein Lagrangian neighborhood theorem.

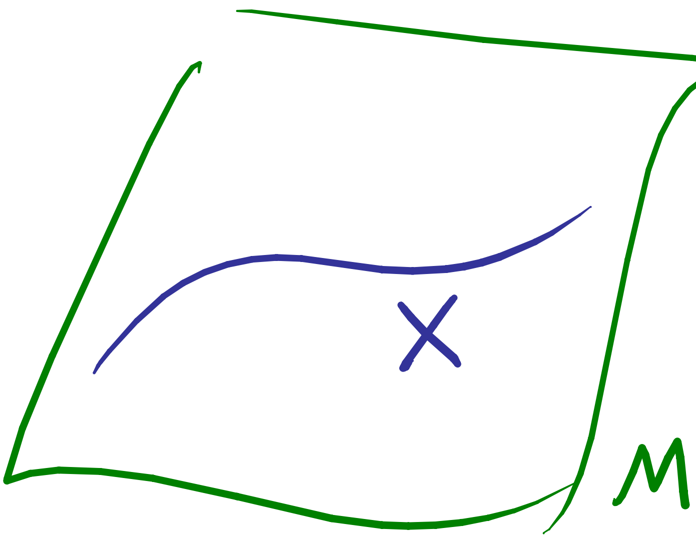
ω_1, ω_2 symplectic on M.

$X \subset M$ Lagrangian for both ω_1, ω_2.

Then $\exists$ tubular neighborhood $\mathcal{U}_1, \mathcal{U}_2$ of X

and $\varphi : (\mathcal{U}_1, \omega_1) \longrightarrow (\mathcal{U}_2, \omega_2)$ (means $\varphi^* \omega_2 = \omega_1$.)

with $\varphi|_X = \mathrm{Id}$.



Want to use relative Moser theorem:

$X \underset{\substack{\text{compact}\\\text{manifold}}}{\subset} M$. ω_0, ω_1 symplectic form on M.

Suppose $\omega_0(x) = \omega_1(x) \quad \forall x \in X$.

Then $\exists$ tubular neighborhood $\mathcal{U}_0, \mathcal{U}_1$ of X and

$\rho : \mathcal{U}_0 \xrightarrow[\text{diffeo.}]{\simeq} \mathcal{U}_1$ with $\rho|_X = \mathrm{Id}_X$

s.t. $\rho^* \omega_1 = \omega_0$.

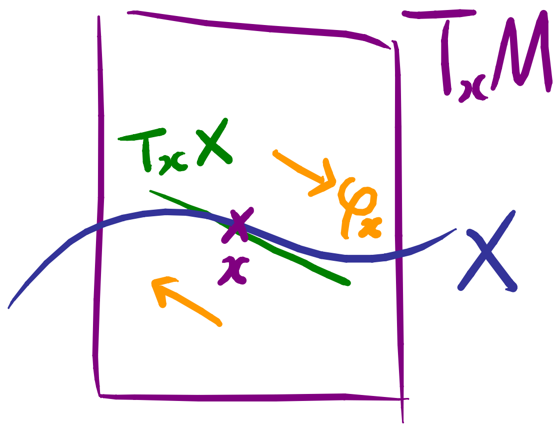
Known: $\iota_X^* \omega_1 = \iota_X^* \omega_2 = 0$.

Need: $\omega_1\Big|_X = \omega_2\Big|_X$.

Strategy

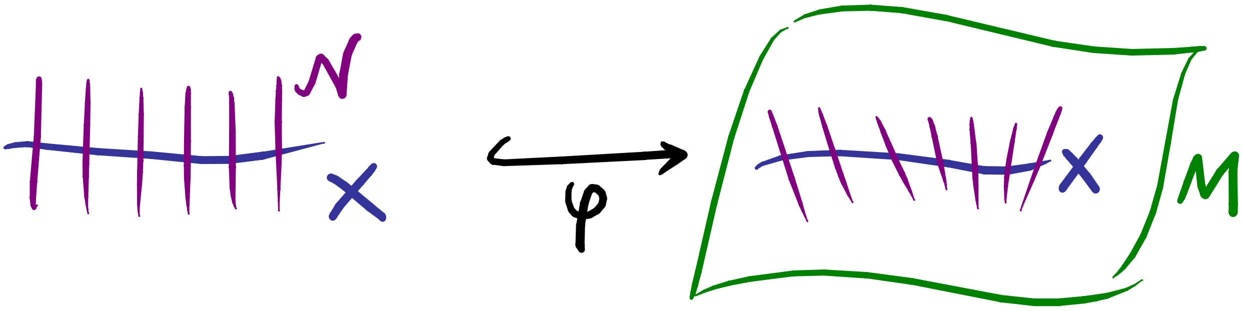
① Show $\exists\, \varphi_x : T_xM \circlearrowleft$ linear such that
(pointwise) smoothly depending on $x \in X$

$$\begin{cases} \varphi_x|_{T_xX} = \mathrm{Id}. \\ \varphi_x^* \omega_2(x) = \omega_1(x). \end{cases}$$



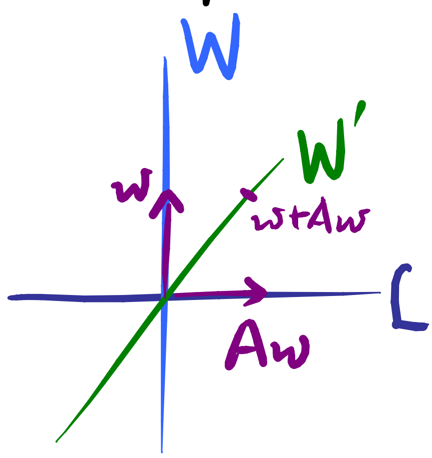
② φ_x integrates to a tubular neighborhood N of X
(integrate) to be $\varphi : N \hookrightarrow M$ such that

$$\varphi|_X = \mathrm{Id} \quad \text{and} \quad \varphi^*\omega_2|_X = \omega_1|_X.$$



Lemma 1: (V, ω) symplectic vector space.

$$L \underset{\text{Lag.}}{<} V.$$



Given $W < V$ with

$$W \oplus L = V,$$

Can construct $W' \underset{\text{Lag.}}{<} V$ with $W' \pitchfork L$.

('modify' W to make it Lagrangian)

Pf: • Want to take

$$W' \triangleq \{ w + Aw : w \in W \}$$

for some $A : W \xrightarrow[\text{linear}]{} L$.

('tilt' W by L.)

(cont.) W' Lag.

$$\Rightarrow \omega(w_1 + Aw_1, w_2 + Aw_2) = 0 \quad \forall w_1, w_2 \in W.$$

$$\| $$

$$\omega(w_1, w_2) + \omega(w_1, Aw_2) + \omega(Aw_1, w_2)$$

$$\Big(L \text{ Lagrangian} \Rightarrow \omega(\underset{L}{\underset{\Uparrow}{Aw_1}}, \underset{L}{\underset{\Uparrow}{Aw_2}}) = 0 \Big)$$

$\therefore$ Want to define $A : W \xrightarrow[\text{linear}]{} L$ such that

$$\omega(w_1, \underset{L}{\underset{\Uparrow}{Aw_2}}) + \omega(\underset{L}{\underset{\Uparrow}{Aw_1}}, w_2) = -\omega(w_1, w_2).$$

$$= -\underset{W^*}{\underset{\Uparrow}{(\imath_{w_1}\omega)}}(w_2).$$

- ω gives
$$\begin{array}{ccc} L & \longrightarrow & W^* . \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v & \longmapsto & \imath_v \omega|_W \end{array}$$

$\cong$: · Same dimension

· injective : $\imath_v \omega|_W = 0 \Rightarrow v \in W^{\perp_\omega}$.

But $v \in L = L^{\perp_\omega}$.

$\therefore$ $v \in (L + W)^{\perp_\omega} = V^{\perp_\omega} \Rightarrow v = 0.$

(cont.) $\therefore\ L \xrightarrow{\sim} W^*.$

$\therefore\ \forall\ w \in W,$

$\exists!\, v \in L$ s.t. $\iota_v \omega = -\frac{1}{2}\omega(w, \cdot) \in W^*.$

Define $A \cdot w \triangleq v.$ (obviously linear.)

Then $\forall\ w_1, w_2 \in W,$

$$\omega(w_1, Aw_2) + \omega(Aw_1, w_2) = -\omega(w_1, w_2).$$

$\therefore\ W' \triangleq \{ w + Aw : w \in W \}$ is Lagrangian. #

e.x. ω_1, ω_2 symplectic on V, and

$L < V$ Lagrangian with respect to both ω_1, ω_2.

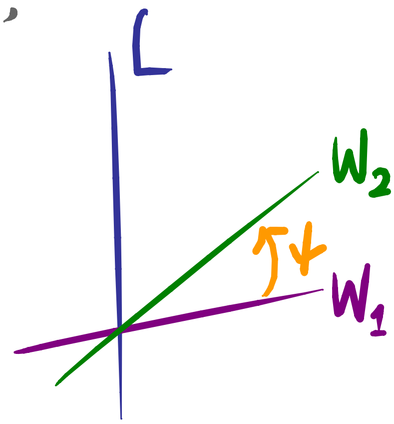
Then every $W < V$ with $W \oplus L = V$

gives $\varphi : (V, \omega_1) \xrightarrow{\simeq} (V, \omega_2)$

with $\varphi|_L = \mathrm{Id}$.

- Lemma 1 $\Rightarrow$ $W_i \underset{\text{Lag.}}{<} V$ with $W_i \pitchfork L$, $i = 1, 2$.

- Have
$$\begin{tikzcd} W_1 \arrow[r, "\iota.\omega_1", "\simeq"'] \arrow[rr, bend right, "\psi"'] & L^* & W_2 \arrow[l, "\iota.\omega_2"', "\simeq"] \end{tikzcd}.$$



Take $\varphi = \mathrm{Id} + \psi : V = L \oplus W_1 \longrightarrow L \oplus W_2 = V$.

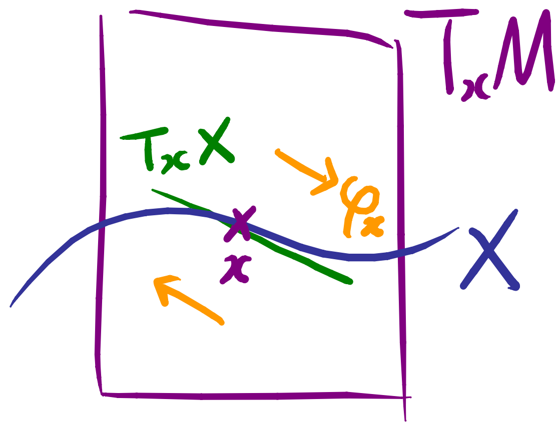
ex. ω_1, ω_2 symplectic on M.

$X \subset M$ Lagrangian for both ω_1, ω_2.

Then $\exists\, \varphi_x : T_xM \circlearrowleft$ linear such that

smoothly depending on $x \in X$

$$\begin{cases} \varphi_x|_{T_xX} = \mathrm{Id}. \\ \varphi_x^* \omega_2(x) = \omega_1(x). \end{cases}$$



Use metric to choose a sub-bundle $W \subset TM|_X$ such that $W \oplus TX = TM|_X$.

Then use the previous exercise.

Whitney extension theorem

$X \underset{\text{submanifold}}{\subset} M.$

Let $\varphi : TM|_X \xrightarrow{\sim} TM|_X$ with $\varphi|_{TX} = \mathrm{Id}.$
(bundle isomorphism)

Then $\exists$ tubular neighborhood $\mathcal{N} \supset X$ and

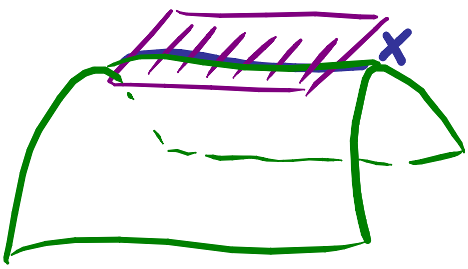
$$\mathcal{N} \xhookrightarrow{h} M \text{ such that}$$

$$h_*|_X = \varphi : TM|_X \circlearrowleft .$$

(Use metric and exponentiate φ to get h.)

e.x. Use Whitney extension theorem to prove Weinstein neighborhood theorem.

(See Step 2.)



e.x. (Weinstein tubular neighborhood theorem)

For $X \underset{\text{Lagrangian}}{\subset} (M, \omega)$,

$\exists$ neighborhood U_0 of $X \subset T^*X$ and

$$\begin{tikzcd}
\varphi : (U_0, \omega_{can}) \arrow[rr] & \arrow[d, phantom, "\circlearrowright"] & (M, \omega) . \\
& X \arrow[ul, bend left] \arrow[ur] &
\end{tikzcd}$$

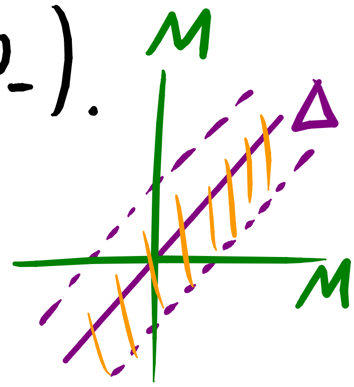
- $NX \underset{\iota, \omega}{\cong} T^*X$.
- Use tubular neighborhood theorem and Weinstein neighborhood theorem.

Application to symplectomorphisms.

$$\Phi : (M, \omega) \circlearrowleft \simeq .$$

$$\Rightarrow \operatorname{graph}(\Phi) \underset{\text{Lagrangian}}{\subset} (M \times M, \omega_-).$$

Also diagonal $\underset{\substack{\parallel \\ \operatorname{graph}(\mathrm{Id})}}{\Delta} \underset{\text{Lagrangian}}{\subset} (M \times M, \omega_-)$.



Weinstein tubular neighborhood theorem

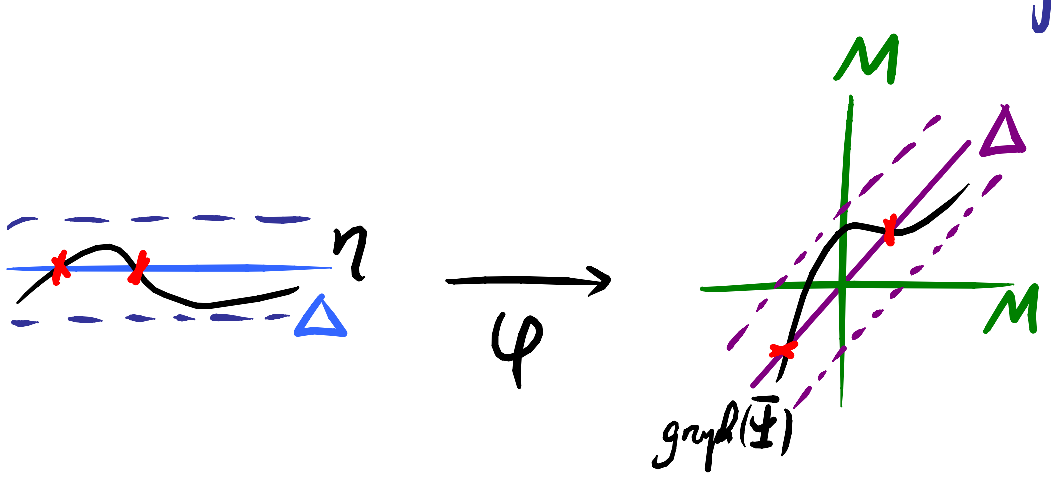
$\Rightarrow$ Tubular neighborhood $\underset{\supset \Delta}{U_0 \subset T^*\Delta}$ and

$$\begin{tikzcd} (U_0, \omega_{can}) \arrow[rr, hook, "\varphi"] & & (M \times M, \omega_-) \\ & \Delta \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

Now if $\Psi \underset{C^1 \text{ closed}}{\sim} \mathrm{Id}$ sufficiently,

$\operatorname{graph}(\Psi) \subset \varphi(\mathcal{U})$ and

$$\varphi^{-1}(\operatorname{graph}(\Psi)) = \operatorname{graph}(\underbrace{\eta}_{\text{closed 1-form on } \Delta \simeq M})$$



$\therefore$ 'Lie algebra' of $\mathrm{Sympl}(M,\omega)$ is $\Omega^1_{\text{closed}}(M)$.

e.x. $\{\text{Symplectic vector fields}\} \underset{\iota_\cdot \omega}{\cong} \Omega^1_{\text{closed}}(M)$.

ex. $\{$Fixed points of $\Psi\} = \{$zeroes of $\eta\}$.

Suppose $H^1(M) = 0$.

Then $\eta = df$.

If M is compact, f has at least two critical points (maximum/minimum points).

$\therefore$ Ψ has at least two fixed points!

Arnold conjecture: (proved)

If $\varphi = \varphi_1$ for φ_t: isotopy generated by X_{H_t}, where H_t is time-dependent with $H_t = H_{t+1}$, then

$$\# \text{ of fixed points of } \varphi \geq \sum_{i=0}^{2n} h^i(M, \mathbb{R}).$$