Examples of Lie groups

Thursday, June 15, 2017 12:11 PM

Motivation: want to describe a smooth family of symmetries.
ex. {rotations on the plane}.

Closely related to geometry: once fix a `structure', want to describe all the changes that preserves the structure.

Lie group: has group structure and smooth structure simultaneously.

Group structure:
Have multiplication, unit, and every element has an inverse.

Smooth structure:
smooth manifold such that multiplication and taking inverse are smooth.

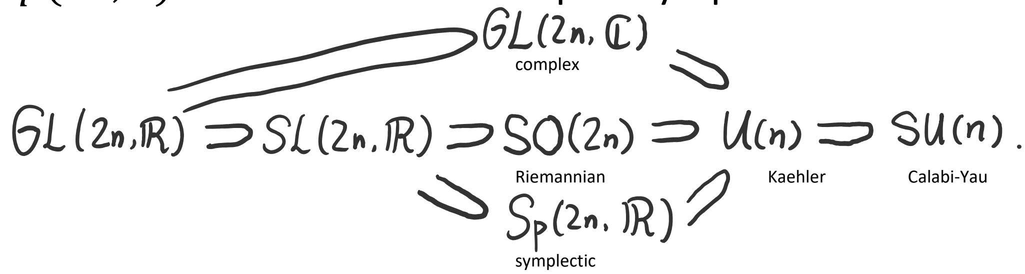
Smooth manifold:
Have covering by charts to open sets in R^n such that transitions are smooth.

Why need smooth manifold?
ex S^1 needs two charts to cover.

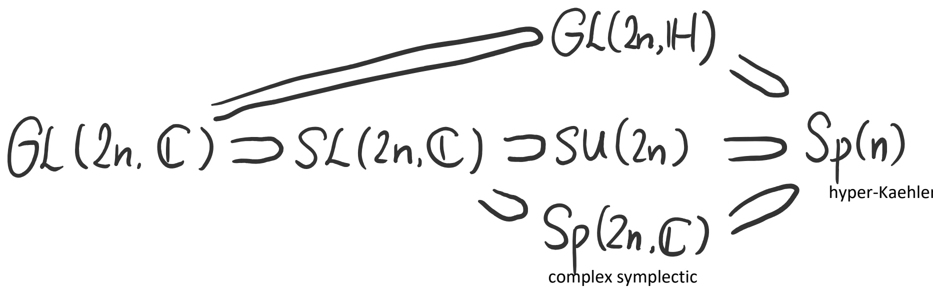
Examples:

1. GL(n,$\mathbb{R}$) (preserve real linear structure) or GL(n,$\mathbb{C}$) (preserve linear complex structure).
2. (Closed subgroups of the above (called matrix Lie groups.)
3. $\mathbb{R} \times \mathbb{R} \times U(1)$ with multiplication defined by $(x_1+x_2, y_1+y_2, e^{i\, x_1\, y_2}\, u_1\, u_2)$.
   This is not a matrix Lie group.
4. SL(n,$\mathbb{R}$) or SL(n,$\mathbb{C}$). Preserves oriented volume.
5. O(n). Preserves linear metric.
6. SO(n). Preserves linear metric and orientation.
7. U(n). Preserves linear Hermitian metric. $U(n) = GL(2n, \mathbb{C}) \cap O(2n)$.
   Important: $h = g + i\omega$. $\omega(v,w) = g(v, Iw)$.
8. SU(n). Preserves complex volume and linear Hermitian metric.

9. $Sp(2n,\mathbb{R})$. Preserve linear symplectic structure. $U(n) = GL(2n,\mathbb{C}) \cap Sp(2n,\mathbb{R})$.
10. $Sp(2n,\mathbb{C})$. Preserve linear complex symplectic structure.



11. $Sp(n) = Sp(2n,\mathbb{C}) \cap U(2n)$. Preserve linear hyperKaehler structure.



12. O(3,1). Preserve linear Lorentz metric.
13. $O(n) \ltimes \mathbb{R}^n$. Rigid motions preserving distance.
14. Heisenberg group

$$\left\{\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R}\right\}.$$

Linear hyperKaehler structure.
Consider $\mathbb{H}^n = (\mathbb{C}^{2n}, J)$ (by $x + y\,j \mapsto (x, y)$, note that j is put in behind so that left multiplication by i is not affected).
$\mathbb{C}^{2n}$ has a standard complex structure I by entriwise multiplication by i.
Additional complex structure: $J \cdot (x, y) := (-\bar{y}, \bar{x})$. Coming from $j \cdot (x + y\,j) = -\bar{y} + \bar{x}\,j$.
Quaternionic linear map: $A \in GL(2n,\mathbb{C})$ with $A \cdot J = J \cdot A$.
(Note that J is only $\mathbb{R}$-linear but not $\mathbb{C}$-linear on $\mathbb{C}^{2n}$. Thus the above equation is defined over $\mathbb{R}$.)
Also have standard holomorphic symplectic form $\omega_{\mathbb{C}}$ and standard Hermitian metric h (conjugate linear in first factor).
Define $K = I \circ J$. Then $I^2 = J^2 = K^2 = IJK = -Id$ (and so $IJ = K, JI = -K$ ...)
In other words we can treat it as a module over the quaternion algebra $\mathbb{H}$ (almost a field except noncommutative).
Have $\omega_{\mathbb{C}}(v, w) = h(Jv, w)$. Thus preserving $\omega_{\mathbb{C}}$ and h implies preserving J.

$Sp(n) = Sp(2n,\mathbb{C}) \cap U(2n) = GL(n,\mathbb{H}) \cap U(2n) = GL(n,\mathbb{H}) \cap Sp(2n,\mathbb{C})$.

Story of quaternions (from Wikipedia)

In 1843, Hamilton knew that complex numbers could be viewed as points in a plane and that they could be added and multiplied together using certain geometric operations. Hamilton sought to find a way to do the same for points in space. Points in space can be represented by their coordinates, which are triples of numbers and have an obvious addition, but Hamilton had difficulty defining the appropriate multiplication.

According to a letter Hamilton wrote later to his son Archibald:
Every morning in the early part of October 1843, on my coming down to breakfast, your brother William Edward and yourself used to ask me: "Well, Papa, can you multiply triples?" Whereto I was always obliged to reply, with a sad shake of the head, "No, I can only add and subtract them."

On October 16, 1843, Hamilton and his wife took a walk along the Royal Canal in Dublin. While they walked across Brougham Bridge (now Broom Bridge), a solution suddenly occurred to him. While he could not "multiply triples", he saw a way to do so for *quadruples*. By using three of the numbers in the quadruple as the points of a coordinate in space, Hamilton could represent points in space by his new system of numbers. He then carved the basic rules for multiplication into the bridge:
$i^2 = j^2 = k^2 = ijk = -1.$

From <https://en.wikipedia.org/wiki/History_of_quaternions>

Note: in the above we mostly concern with linear structures. We easily get to `infinite-dimensional Lie groups' if we consider geometric structures on a manifold.
ex. Diffeo(M), Sympl(M),…
The matrix groups can be treated as structure groups of the tangent space.
ex. real manifold, complex manifold, symplectic manifold, Riemannian manifold, Kaehler manifold, Calabi-Yau manifold, hyperKaehler manifold…
WARNING: we have skipped INTEGRABILITY CONDITIONS on geometric structures.

Exercises. (Section 1.6)

3. $Sp(2,\mathbb{R})=SL(2,\mathbb{R})$;
   $Sp(2,\mathbb{C})=SL(2,\mathbb{C})$;

Sp(1)=SU(2).

4. Let a be irrational and
$G = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix} : t \in \mathbb{R} \right\}.$
Show that the closure in $GL(2, \mathbb{C})$ gives $\left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix} : \theta, \phi \in \mathbb{R} \right\}.$