Closed subgroup theorem

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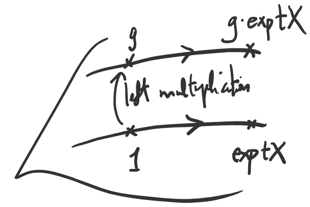
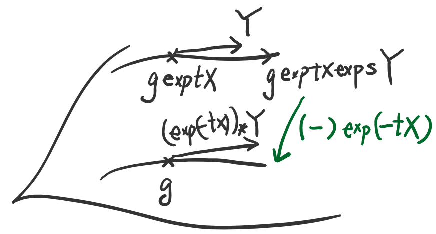
Ref: Lee - Introduction to Smooth Manifolds Ch. 19, 20.

Lie(G) := $T_1 G \cong$ {left-G-invariant vector fields}.
Integrating along $X \in Lie(G)$ gets a
one-parameter subgroup of diffeomorphisms $\mathbf{exp^t}\, \boldsymbol{X} = \exp tX$.
($\exp(s+t)X = \exp sX \circ \exp tX$.)
$X \in \mathrm{Lie}(G)$ is complete: walk a bit, and left multiplication by the endpoint.
Abuse of notation: under the diffeomorphism $\exp tX$, denote $1 \mapsto \exp tX \in G$.
This defines $\mathbf{exp}\colon \mathfrak{g} \to \boldsymbol{G}$. (Don't even use metric!)
Since X is left-invariant, the diffeomorphism $\exp^{\mathrm{t}} X$ is $g \mapsto g \exp tX$ (right multiplication.)
$d\exp|_0 = \mathrm{Id}$. Hence exp is a local diffeomorphism.

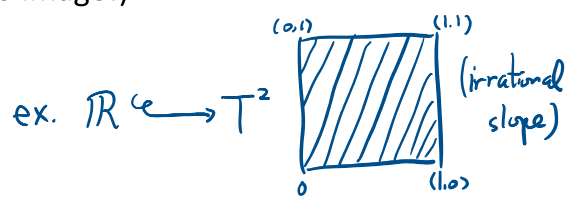


Lie bracket on Lie(G): $[X, Y] = L_X Y$ (Lie derivative).
Lie deriative satisfies $[X,Y] = -[Y,X]$ and Jacobi identity.
$L_V W$ is still left-invariant:
$L_X Y|_g = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} g \exp tX \exp sY \exp(-tX)$. Thus
$h \cdot \left(L_X Y|_g\right) = L_X Y|_{hg}$.



Lie(G) is a **Lie algebra**: a vector space with $[-,-]$.
Lie subalgebra: subspace closed under $[-,-]$.

Lie subgroup H: Image of an injective group homomorphism to G which is an immersion. (NOTE: NEED NOT EMBEDDED, that is, may not be homeomorphism to image!)



Theorem:
Subalgebra of Lie(G) <-> connected Lie subgroup of G.

<- is trivial: take tangent space at 1. Closed under Lie bracket since H is a submanifold near 1.
-> follows from the **Frobenius theorem** in differential topology:
A sub-bundle of TM whose sheaf of local sections are closed under $[-,-]$ integrates to a foliation.
Then take the leaf containing 1. It is closed under multiplication:
suppose $g, h \in \mathrm{leaf}_1$. $g \cdot h = L_g \cdot h \in L_g \cdot \mathrm{leaf}_1 = \mathrm{leaf}_g = \mathrm{leaf}_1$.

(**Foliation**: a collection of disjoint connected immersed submanifolds (called leafs) whose union is the whole space M, and can take local coordinates $x_1, \ldots, x_n$ of M such that the leafs are given by taking $x_{k+1}, \ldots, x_n$ to be constants.)

Closed subgroup theorem:
A closed subgroup H of G must be a submanifold (that is embedded).
(Note: don't need H is Lie subgroup in the condition.)

Proof:
Need to restricts charts of G to charts of H at all $h \in H$. Consider an open set U of $\mathfrak{g}$ where exp is a diffeomorphism.
Want to argue $h \cdot \exp$ restricted to $U \cap \mathfrak{h}$ provides a chart of H. Need to define $\mathfrak{h}$!
$\mathfrak{h} \coloneqq \{X \in \mathfrak{g}: \exp tX \in H \text{ for all } t \in \mathbb{R}\}$.
$\mathfrak{h} \subset \mathfrak{g}$ is a vector subspace: $0 \in \mathfrak{h}$. Closed under scaling.
$X + Y \in \mathfrak{h}$ if X and Y are: consider $\exp t(X+Y)$.
TRICKY: $\exp t(X+Y) \neq (\exp tX)(\exp tY)$!
For t small,
$(\exp tX)(\exp tY) = \exp(\phi(t))$ for some smooth path ϕ. ($\phi(0) = 0$.)
Take $\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}$, get $\phi'(0) = X + Y$. Thus $\phi = t(X+Y) + t^2 Z(t)$.
Replace t by $\frac{t}{n}$:
$\left(\left(\exp\frac{tX}{n}\right)\left(\exp\frac{tY}{n}\right)\right)^n = \exp\left(t(X+Y) + \frac{t^2}{n} Z\left(\frac{t}{n}\right)\right)$, and hence

$$\lim_{n\to\infty}\left(\left(\exp\frac{tX}{n}\right)\left(\exp\frac{tY}{n}\right)\right)^n = \exp t(X+Y)$$

Since H is closed, LHS belongs to H.
Thus $\mathfrak{h} \subset \mathfrak{g}$ is a vector subspace.

Need to take U sufficiently small such that $\exp(U \cap \mathfrak{h}) = (\exp U) \cap H$. (Always have $\exp(U \cap \mathfrak{h}) \subset (\exp U) \cap H$.) Then $h \cdot \exp$ restricted to $U \cap \mathfrak{h}$ provides a chart of H around h.
Assume such U does not exist. Then have a sequence of points in H converging to 1 but not in $\exp(U \cap \mathfrak{h})$.
Take a meric on $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. Then the points can be written as $(\exp a_i)(\exp b_i)$ for $a_i \in \mathfrak{h}$ and $b_i \in \mathfrak{h}^{\perp}$. NOTE THAT $\exp b_i \in H$.
b_i are normalized to points on the unit sphere. Take a convergent subsequence and denote its limit by $v \in \mathfrak{h}^{\perp}$. For any $t \in \mathbb{R}$, tv is a limit of $\{t_i b_i\}$ for some t_i. $\exp t_i b_i \in H$, and hence $\exp tv \in H$ since H is closed! Then $v \in \mathfrak{h}$ by definition of $\mathfrak{h}$, a contradiction!

Lie homomorphism $G \to H$:
smooth group homomorphism.

Theorem:
Continuous homomorphism $G \to H$ is automatically smooth!

Proof:
First, any continuous homomorphism $\gamma: \mathbb{R} \to H$ is $\gamma(t) = \exp tX$ for some X (and hence smooth):
For t_0 small, $\gamma(t_0) = \exp t_0 X$ for some X. $\gamma(t_0) = \gamma\left(\frac{t_0}{k}\right)^k = \exp\left(\frac{t_0 X}{k}\right)^k$ and hence
$\gamma\left(\frac{t_0}{k}\right)^{\square} = \exp\left(\frac{t_0 X}{k}\right)^{\square}$ (since exp is a diffeomorphism in the small region).
Then $\gamma\left(\frac{p\, t_0}{k}\right)^{\square} = \exp\left(\frac{p\, t_0 X}{k}\right)^{\square}$ for all p, k. By continuity $\gamma(t) = \exp tX$.

Now consider $\Phi: G \to H$, $1_G \mapsto 1_H$. Use charts provided by exp to understand the map.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\Phi"'] & H \end{tikzcd}$$

$\Phi \circ \exp_G(X) = \exp_H \circ\, \phi(X)$. ϕ is a priori only defined near $X = 0$.
For each X, $\Phi \circ \exp_G(tX)$ gives a continuous homomorphism $\mathbb{R} \to H$. From above $\Phi \circ \exp_G(tX) = \exp_H tY$. Such Y is unique since $\exp_H$ is a local diffeomorphism.
We define $\phi(X) = Y$. Thus $\phi: \mathfrak{g} \to \mathfrak{h}$.
Suffice to prove ϕ is linear, and hence Φ is smooth around 1_G.
Then $\Phi = L_g \circ \Phi \circ L_{g^{-1}}$ (since it is homomorphism) is smooth around g.

ϕ is linear:
$\Phi \circ \exp_G(t\,sX) = \exp_H t\,\phi(sX) = \exp_H\ t\,s\phi(X)$. Thus $\phi(sX) = s\phi(X)$.

$$\exp_H \phi(X+Y) = \Phi \circ \exp_G(X+Y) = \Phi\left(\lim_{n\to\infty}\left(\exp_G\left(\frac{X}{n}\right)\exp_G\left(\frac{Y}{n}\right)\right)^n\right)$$
$$= \lim_{n\to\infty} \Phi\left(\exp_G\left(\frac{X}{n}\right)\exp_G\left(\frac{Y}{n}\right)\right)^n \quad (\Phi \text{ is continuous})$$
$$= \lim_{n\to\infty}\left(\left(\Phi \circ \exp_G \frac{X}{n}\right)\left(\Phi \circ \exp_G \frac{Y}{n}\right)\right)^n \quad (\Phi \text{ is homomorphism})$$
$$= \lim_{n\to\infty}\left(\left(\exp_H \phi\left(\frac{X}{n}\right)\right)\left(\exp_H \phi\left(\frac{Y}{n}\right)\right)\right)^n$$
$$= \lim_{n\to\infty}\left(\left(\exp_H \frac{\phi(X)}{n}\right)\left(\exp_H \frac{\phi(Y)}{n}\right)\right)^n$$
$$= \exp_H\left(\phi(X) + \phi(Y)\right).$$

Thus $\phi(X+Y) = \phi(X) + \phi(Y)$ for X, Y small enough.
By rescaling this is true for all X, Y.

Exercises. (Section 2.6)

5. Show that
$\exp\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} e^a & b\,\frac{e^a - e^d}{a-d} \\ 0 & e^d \end{pmatrix}$ (where the right hand side is defined by taking limit when $a = d$.

8. Consider $X = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Compute e^{tX} and e^{tY} by diagonalization. Visualize the curves $e^{tX} \cdot v$ and $e^{tY} \cdot v$ for $v \neq 0$.