Adjoint actions

Tuesday, February 6, 2018 9:30 AM

$$\mathfrak{gl}(2n,\mathbb{R}) \supset \mathfrak{sl}(2n,\mathbb{R}) \supset \mathfrak{so}(2n,\mathbb{R}),\ \mathfrak{gl}(n,\mathbb{C}),\ \mathfrak{sp}(2n,\mathbb{R}) \supset \mathfrak{u}(n) \supset \mathfrak{su}(n)$$

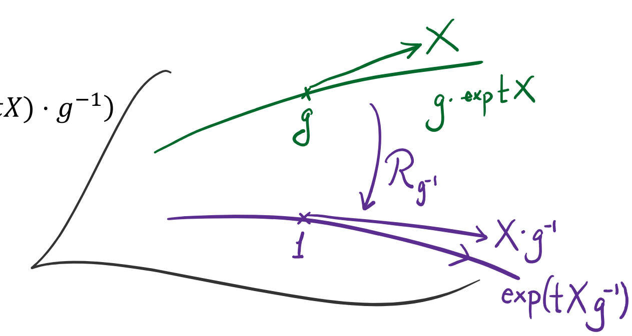
Lie(G):
1. $\mathfrak{gl}(2n,\mathbb{R})$: all matrices.
2. $\mathfrak{sl}(2n,\mathbb{R})$: $\mathrm{tr}(X)=0$. Obtained by taking $\frac{d}{dt}\Big|_{t=0}$ on $\det A(t)=1$ where $A(0)=\mathrm{Id}$ and $\frac{d}{dt}\Big|_{t=0} A(t)=X$.
3. $\mathfrak{so}(2n,\mathbb{R})$: $X=-X^T$.
4. $\mathfrak{gl}(2n,\mathbb{C})$: all complex matrices.
5. $\mathfrak{sp}(2n,\mathbb{C})$:
$$\left(\begin{pmatrix}0 & -Id\\ Id & 0\end{pmatrix}X\right)^T=\begin{pmatrix}0 & -Id\\ Id & 0\end{pmatrix}X.$$
6. $\mathfrak{u}(n)$: $X=-X^*$.
7. $\mathfrak{su}(n)$: $X=-X^*$ and $\mathrm{tr}(X)=0$.

Lie derivative:
$$ad(X)\cdot Y=[X,Y]=\frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\exp tX \exp sY \exp(-tX)$$
For $\mathfrak{gl}(2n,\mathbb{R})$: $XY-YX$.

Adjoint action on $\mathfrak{g}=\mathrm{Lie}(G)=T_1G$ in general: $Ad(g)\cdot Y=R_{g^{-1}}\cdot Y|_g$.
For $\mathfrak{gl}(2n,\mathbb{R})$: gYg^{-1}.

$ad(X)=\frac{d}{dt}\Big|_{t=0}Ad(\exp tX)$.
$Ad(g)$ can be understood as right multiplication by g^{-1} on (left-invariant) vector fields. Denote $Ad(g)\cdot Y=Y\cdot g^{-1}$. Obvious that
$Ad(g\cdot h)=Ad(g)\circ Ad(h)$.

$Ad(g)\cdot[X,Y]=[Ad(g)\cdot X, Ad(g)\cdot Y]$:
$$Ad(g)\cdot[X,Y]=\frac{d}{dt}\Big|_{t=0}(Y\cdot\exp -tX)\cdot g^{-1}=\frac{d}{dt}\Big|_{t=0}(Y\cdot g^{-1})\cdot(g\cdot(\exp -tX)\cdot g^{-1})$$
$$=\frac{d}{dt}\Big|_{t=0}(Y\cdot g^{-1})\cdot(\exp(-t\,X\cdot g^{-1}))=[Ad(g)\cdot X, Ad(g)\cdot Y]$$



Killing form in general: $\langle X_1,X_2\rangle=\mathrm{tr}(ad(X_1)\circ ad(X_2)$ on $\mathfrak{g})$.
For $\mathfrak{gl}(n,\mathbb{R})$: $2n\,\mathrm{tr}(X_1X_2)-2\,\mathrm{tr}(X_1)\mathrm{tr}(X_2)$.
Thus it is $2n\,\mathrm{tr}(X_1X_2)$ for $\mathfrak{sl}(n,\mathbb{R})$ which is positive definite on symmetric matrices and negative definite on skew-symmetric matrices (and hence indefinite on $\mathfrak{sl}(n,\mathbb{R})$).
Can verify by using the basis e_{ij} of $\mathfrak{gl}(n,\mathbb{R})$.
$\mathrm{tr}(X\,\cdot)=n\,\mathrm{tr}(X)=\mathrm{tr}(\cdot\,X)$; $\mathrm{tr}(X_1\cdot(-)\cdot X_2)=(\mathrm{tr}\,X_1)(\mathrm{tr}\,X_2)$.
For $\mathfrak{su}(n)$: $2n\,\mathrm{tr}(X_1X_2)$. Verify in a similar way. This is negative definite since $X_2=-X_2^*$.

Adjoint action always preserve the Killing form:
$\mathrm{tr}(ad(X_1)\circ ad(X_2))=\mathrm{tr}(ad(Ad(g)\cdot X_1)\circ ad(Ad(g)\cdot X_2))$.
$ad(Ad(g)\cdot X_1)\circ ad(Ad(g)\cdot X_2)\cdot Y=\big[Ad(g)\cdot X_1,[Ad(g)\cdot X_2,Y]\big]=Ad(g)\big[X_1,[X_2,Ad(g^{-1})\cdot Y]\big]$
$=Ad(g)\circ ad(X_1)\circ ad(X_2)\circ Ad(g)^{-1}$
whose trace equals to $\mathrm{tr}(ad(X_1)\circ ad(X_2))$.

Examples of Lie homomorphisms:

1. det: $GL(n,\mathbb{C}) \to \mathbb{C}^\times$. Ker $= SL(n,\mathbb{C})$.
2. $\mathbb{R} \to \mathrm{SO}(2)$ rotation by $\theta \in \mathbb{R}$.
3. $Ad: G \to GL(\mathfrak{g})$. Correspondingly $ad: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism.
   Moreover $Ad(g): \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism.
4. $\mathrm{SU}(2) \to \mathrm{SO}(3)$ by acting on $\mathfrak{su}(2)$ (space of skew-Hermitian matrices) by gXg^{-1}. The adjoint action preserves the Killing form which is just the standard metric (up to scaling).
   $SU(2) \cong \mathbb{S}^3$: Identify $\alpha + \beta j \in \mathbb{H}$ as $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$. Then $SU(2) = \{|\alpha + \beta j|^2 = 1\} \subset \mathbb{H}$.
   Conjugate transpose is quaternionic conjugation.
   Matrix multiplication is quaternionic multiplication.
   $\mathfrak{su}(2)$ is identified as Im $\mathbb{H}$.
   The adjoint action is $uxu^{-1} = ux\bar{u}$.
   The homomorphism is 2:1. Consider preimage of Id: $uxu^{-1} = x$ for all $x \in \mathrm{Im}\,\mathbb{H}$. Then $u \in \mathbb{R}$. But $|u|^2 = 1$, and hence $u = \pm 1$. Indeed $\pm u$ maps to the same rotation in $SO(3)$. Hence $SO(3) = \mathbb{S}^3/\pm = \mathbb{RP}^3$.
   Note that $ux\bar{u}$ fixes u and $\bar{u}$, and hence $\frac{u-\bar{u}}{2} \in \mathrm{Im}\,\mathbb{H}$. This is the axis of rotation. Normalize $\frac{u-\bar{u}}{2}$ to h. Then $u = \cos\theta/2 + h\sin\theta/2$ for some θ. Then $ux\bar{u}$ is rotation by θ. For instance take $h = i$, then $ux\bar{u} = e^{i\theta/2}\,(ai + bj + ck)\,e^{-i\theta/2} = ai + e^{i\theta}\,(bj + ck)$ which is rotating the {j,k}-plane by θ.
   Thus the homomorphism is surjective.
5. Given a Lie homomorphism $\Phi: G \to H$, have the tangent map $\phi: \mathfrak{g} \to \mathfrak{h}$ with $\Phi \circ \exp_G(X) = \exp_H \circ\, \phi(X)$.
   Consider $\Phi: \mathrm{SU}(2) \to \mathrm{SO}(3)$, $g \mapsto Ad(g)$. $\phi(X) = ad(X) = [X, -]$.
   Explicit: take the basis $E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, E_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, E_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\mathfrak{su}(2)$. Then
   $[E_1, E_2] = E_3, [E_2, E_3] = E_1, [E_3, E_1] = E_2$. Thus
   $$ad(E_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, ad(E_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, ad(E_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercises. (Section 3.9)

10. Show that there is a linear isomorphism $\phi: \mathfrak{su}(2) \to \mathbb{R}^3$ such that $\phi([X,Y]) = \phi(X) \times \phi(Y)$ (the cross product for $\mathbb{R}^3$).
11. Show that $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$ are not isomorphic Lie algebras.