Have $\exp : \mathfrak{gl}(n,\mathbb{C}) \to GL(n,\mathbb{C})$. The defining property is
$\frac{d}{dt}\exp tX = X\Big|_{\exp tX} = (L_{(\exp tX)})_* X\Big|_0 = (\exp tX)\cdot X$ with $\exp 0 = 1_{GL(n)}$.
(Since $GL(n)$ is an open subset of $\mathrm{Mat}_{n\times n}$, tangent space at any point is identified with $\mathrm{Mat}_{n\times n}$.)

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$$
satisfies the defining properties.
Indeed it absolutely converges under the standard complete **Hilbert-Schmidt matrix norm**.
(Note that $|X^n| \le |X|^n$.)

From the definition,
1. $\left(e^X\right)^* = e^{X^*}$.
2. $e^{X+Y} = e^X e^Y = e^Y e^X$ if $XY = YX$.
3. $|e^X| \le e^{|X|}$.
4. $e^{CXC^{-1}} = Ce^X C^{-1}$.

To **compute** e^X, take Jordan canonical form $S+N$ where S is diagonal, N is nilpotent.
Then $e^{C(S+N)C^{-1}} = Ce^S e^N C^{-1}$.

exp is invertible in a small neighborhood of 0. Call its inverse to be log. Want to have
$$\log A = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}(A-I)^m}{m} \text{ for } A \in GL(n).$$

The above series converges **for complex numbers** in the disc $|z-1|<1$.
$e^{\log z} = z$ holds since it holds on the interval (0,2) (identity theorem).
To talk about $\log e^w = w$, need $|e^w - 1| < 1$.
$$|e^w - 1| = \left| w + \frac{w^2}{2!} + \cdots \right| \le |w| + \frac{|w|^2}{2!} + \cdots = e^{|w|} - 1 < 1$$
if $|w| < \log 2$. Then by identity theorem $\log e^w = w$.

Now consider $\{|A-I|<1\}$. Then the above series converges.
For diagonal matrix A, it is obvious that $e^{\log A} = A$. Since **every matrix is the limit of diagonal matrices**, this still holds for general matrices.
Similarly $\log e^X = X$, if $|e^X - I| < 1$. This is ensured by $|X| < \log 2$.

Theorem:

$\exp : \mathfrak{gl}(n,\mathbb{C}) \to GL(n,\mathbb{C})$ is surjective.

Proof:

Take Jordan canonical form. For each block $J = \lambda I + N = \lambda\left(I + \frac{N}{\lambda}\right)$, consider

$$\log J := (\log\lambda) I + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\left(\frac{N}{\lambda}\right)^m}{m}$$

which makes sense since $(N)^m = 0$ for m large enough.

Then $\exp \log J = J$: cannot use the previous argument since $\left|\frac{N}{\lambda}\right|$ can be big.

Consider $J_t = \lambda\left(I + \frac{t\,N}{\lambda}\right)$. By previous result $\exp\log J_t = J_t$ for t small. Also both LHS and RHS are analytic in t (indeed polynomial). Hence true for all t.

Note that exp is NOT injective: $\log\lambda$ has different branches. So exp is invertible only in a small neighborhood (so that we can consistently take the branch $\log 1 = 0$.)

Note: $\exp: \mathfrak{gl}(n,\mathbb{R}) \to GL(n,\mathbb{R})$ is NOT surjective. (Nor $\mathfrak{sl}(n,\mathbb{R}) \to SL(n,\mathbb{R})$.)

For instance take $\begin{pmatrix} -4 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$. There is no real square root which must have four eigenvalues $\pm 2i, \pm\frac{i}{2}$ (invariant under conjagation).

Even though exp is not surjective,

every g can be written as $\exp X_1 \ldots \exp X_k$ for some $X_1, \ldots, X_k$ if G is connected:

The set of all such elements is both open and closed: exp is a local diffeomorphism around 0.

In particular the **image of the exponential map may not be a subgroup**: the image generates the whole connected Lie group, but some elements may not lie in the image.

Also **exp may not be an open map** (even for compact Lie group): consider $SU(2)$ and $\begin{pmatrix} \pi i & 0 \\ 0 & -\pi i \end{pmatrix} \in \mathfrak{su}(2)$ which is mapped to -1. Open neighborhood: two distinct eigenvalues (near πi and $-\pi i$) with eigen-directions close to (1,0) and (0,1).

Image under exp: same eigen-directions, exp the eigenvalues.

But near $-1 \in SU(2)$, have $-\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with eigenvectors (1,i) and (1,-i). They are not contained in the image and hence image is not open.

(The problem is that exp maps the whole unit sphere in $\mathfrak{su}(2)$ to a point.)

For compact Riemannian manifold, exp is surjective.

Exercises. (Section 2.6)

4. Using the Jordan canonical form, show that every complex square matrix is the limit of a sequence of diagonalizable matrices. (For instance consider the vectors $e_1 + \epsilon\, e_2 + \cdots + \epsilon^{k-1} e_k$.)

10. Using log ⬚, show that for $X \in \mathfrak{gl}(n, \mathbb{C})$,

$$\lim_{m \to \infty} \left(1 + \frac{X}{m}\right)^m = e^X.$$