

Simple Lie algebra: $\dim \mathfrak{g} \geq 2$ and the only ideals are $\{0\}$ and itself.

Have classification by Dynkin diagram.

For instance, $\mathfrak{sl}(2, \mathbb{C})$ is simple: take the basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H.$$

Suppose $Z = aX + bY + cH$ is in the ideal. $[X, Z] = bH - 2cX$. $[X, [X, Z]] = -2bX$. If $b \neq 0$, then X is in the ideal. Then $[X, Y] = H, [H, Y] = -2Y$ imply that H and Y are also in the ideal which has to be $\mathfrak{sl}(2, \mathbb{C})$.

The case $b = 0$: do the same argument for $[Y, [Y, Z]]$, then the ideal is $\mathfrak{sl}(2, \mathbb{C})$ unless $a = 0$.

The case $a = b = 0$: $[H, X] = 2X, [H, Y] = -2Y$ implies the ideal is $\mathfrak{sl}(2, \mathbb{C})$ unless $c = 0$. The ideal is $\{0\}$ if $a = b = c = 0$.

Commutator ideal: $[\mathfrak{g}, \mathfrak{g}] = \text{Span}\{[X, Y] : X, Y \in \mathfrak{g}\}$.

An ideal is in particular a Lie subalgebra.

Keep on taking commutator ideals, get $\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_{j+1} = [\mathfrak{g}_j, \mathfrak{g}_j] \subset \mathfrak{g}_j, \dots$ called derived series.

Solvable: $\mathfrak{g}_j = \{0\}$ for some j .

(If \mathfrak{g} is simple, $\mathfrak{g}_j = \mathfrak{g}$ for all j .)

Levi decomposition: any Lie algebra is the semi-direct product of a solvable ideal and a semisimple subalgebra.

Similar concept: $\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^{j+1} = [\mathfrak{g}, \mathfrak{g}^j] = [\mathfrak{g}, [\mathfrak{g}, [\dots, \mathfrak{g}]]] \subset \mathfrak{g}^j$. Sequence of ideals called **lower central series**.

Nilpotent: $\mathfrak{g}^j = \{0\}$ for some j .

$\mathfrak{g}_j \subset \mathfrak{g}^j$. Hence nilpotent implies solvable.

For instance, the Lie algebra of nilpotent upper triangular matrices \mathfrak{n} is nilpotent (and hence solvable).

Can take the basis $E_{i,j}, j > i$. Then

$$[E_{i,j}, E_{k,l}] = 0 \text{ if } j \neq k \text{ and } = E_{i,l} \text{ if } j = k.$$

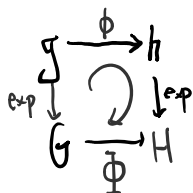
\mathfrak{n}^k is spanned by $E_{i,j}$ for $j - i > k$ and hence $\mathfrak{n}^{n-1} = \{0\}$.

The Lie algebra of upper triangular matrices \mathfrak{u} is solvable but not nilpotent.

Can take the basis $E_{i,i}, E_{i,i+1}, E_{i,i+2}, \dots, E_{1,n}$. Similar as above and $[E_{i,i}, E_{j,j}] = 0$.

$\mathfrak{u}_1 = \mathfrak{u}^1 = \mathfrak{n}$. So $\mathfrak{u}_n = 0$. But $\mathfrak{u}^i = \mathfrak{u}^1$ for all $i \geq 1$.

Recall that any Lie group homomorphism $\Phi: G \rightarrow H$ corresponds to a linear map $\phi: \mathfrak{g} \rightarrow \mathfrak{h}$. (In the proof that a continuous homomorphism is smooth.)



Since $\Phi(gxg^{-1}) = \Phi(g)\Phi(x)\Phi(g)^{-1}$, $\phi \circ \text{Ad}_G(g) = \text{Ad}_H(\Phi(g)) \circ \phi$.

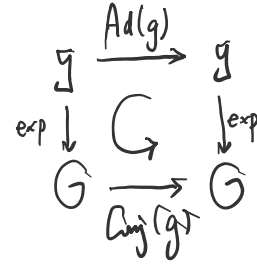
Then $\phi \circ \text{ad}_g(X) = \text{ad}_h(\phi(X)) \circ \phi$, that is $\phi([X, Y]) = [\phi(X), \phi(Y)]$.

Thus **Lie group homomorphism corresponds to a Lie algebra homomorphism.**

In particular take $\Phi = \text{Ad}: G \rightarrow GL(\mathfrak{g})$. It corresponds to the Lie algebra homomorphism $\phi = \text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$. Hence

$$\text{Ad}(\exp X) = \exp(\text{ad } X) = \sum_{n=0}^{\infty} \frac{(\text{ad } X)^n}{n!}.$$

Thus **$\exp X$ commutes with $\exp tY$ if $[X, Y] = \mathbf{0}$** using that $\exp \circ \text{Ad}(g) = \text{Conj}(g) \circ \exp$ (because conjugation pushes a left invariant vector field to a left invariant one).



Take $G = GL(n, \mathbb{C})$. Then

$$e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{(\text{ad } X)^n \cdot Y}{n!} = \sum_{n=0}^{\infty} \frac{[X, [X, \dots, [X, Y]]]}{n!}.$$

If G is connected, then Φ is determined by ϕ :

Recall that any element $g = e^{X_1} \cdot \dots \cdot e^{X_n}$ if G is connected. $\Phi(g) = e^{\phi(X_1)} \cdot \dots \cdot e^{\phi(X_n)}$.

In particular **\mathfrak{g} commutative implies (connected) G commutative.** ($\text{ad} = \text{Id}$, which is the tangent map for both Ad and Id .)

Any real Lie algebra (whose underlying vector space is over \mathbb{R}) can be complexified. For instance, $\mathfrak{gl}(n, \mathbb{R}) \otimes \mathbb{C} \cong \mathfrak{gl}(n, \mathbb{C}) \cong \mathfrak{u}(n) \otimes \mathbb{C}$.

Thm:

Suppose G is connected. There exists a one-to-one correspondence between connected normal Lie subgroups and ideal of Lie algebras.

Recall: H is normal means $GHG^{-1} = H$.

Lie subgroup means a subgroup which is also an immersed submanifold.

Proof:

Already have correspondence between connected Lie subgroups and Lie subalgebras (Frobenius theorem). (Don't need G connected for this.)

\Rightarrow) For $X \in \mathfrak{g}, Y \in \mathfrak{h}$, $\exp tX \exp sY \exp -tX \in H$. Taking derivatives gives $[X, Y] \in \mathfrak{h}$.

\Leftarrow) $\text{Ad}(e^X) \cdot Y = \sum_{n=0}^{\infty} \frac{(\text{ad } X)^n \cdot Y}{n!} \in \mathfrak{h}$.

Since $\exp \circ \text{Ad}(e^X) = \text{Conj}(e^X) \circ \exp$, $e^X \cdot e^Y \cdot e^{-X} \in H$.

Since H is connected, any $h = e^{Y_1} \cdot \dots \cdot e^{Y_n}$. Hence $e^X \cdot h \cdot e^{-X} \in H$.

Since G is connected, any $g = e^{X_1} \cdot \dots \cdot e^{X_n}$. Hence $g \cdot h \cdot g^{-1} \in H$.

Prop:

If $G \rightarrow H$ induces an isomorphism $\mathfrak{g} \rightarrow \mathfrak{h}$ and H is simply connected, then $G \rightarrow H$ is an isomorphism:

$G \rightarrow H$ induces an isomorphism on the tangent spaces together with left multiplications implies $G \rightarrow H$ must

be a covering map. If H is simply connected, then $G \rightarrow H$ must be a diffeomorphism.

Thm:

If G is simply connected, then any $\mathfrak{g} \rightarrow \mathfrak{h}$ integrates to $G \rightarrow H$.

Proof:

Consider the graph of $\mathfrak{g} \rightarrow \mathfrak{h}$ which is a subalgebra in $\mathfrak{g} \times \mathfrak{h}$. It corresponds to a Lie subgroup Gr in $G \times H$.

The first projection $Gr \rightarrow G$ is an isomorphism and hence invertible:

the corresponding map $\mathfrak{g}r \rightarrow \mathfrak{g}$ is an isomorphism, and G is simply connected.

Thus have $G \rightarrow Gr$. Compose it with the second projection $Gr \rightarrow H$ to get $G \rightarrow H$.

Conclusion:

| | | | |
|------------------------------------|-------------------|--|----------------------------|
| $X \in \mathfrak{g}$ | | all $g = e^{X_1} \dots e^{X_n} \in G$ | if G is connected |
| Lie subalgebra | \leftrightarrow | connected Lie subgroup (which can be immersed) | |
| ideal | \leftrightarrow | connected normal Lie subgroup | if G is connected |
| $\exp(\text{ad}(X))$ | \leftrightarrow | $\text{Ad}(\exp X)$ | |
| $[\mathfrak{g}, \mathfrak{g}] = 0$ | \leftrightarrow | $gh=hg$ | if G is connected |
| group homomorphism | \leftrightarrow | algebra homomorphism | if G is simply connected |

Exercises. (Section 3.9)

4. Give an example of $G \subset GL(n, \mathbb{C})$ and $X \in \mathfrak{gl}(n, \mathbb{C})$ such that $e^X \in G$ but $X \notin \text{Lie}(G)$.

21. Let $A \in SL(n, \mathbb{R})$ which has an eigenvalue in $\mathbb{C} - \mathbb{R}$. Show that

$$A = C \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} C^{-1}$$

for some invertible matrix C .