Simple Lie algebra: dim \( g \geq 2 \) and the only ideals are \{0\} and itself. Have classification by Dynkin diagram.

For instance, \( \mathfrak{sl}(2, \mathbb{C}) \) is simple: take the basis
\[
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Suppose \( Z = aX + bY + cH \) is in the ideal. \([X, Z] = bH - 2cX, [X, [X, Z]] = -2bX. \) If \( b \neq 0 \), then \( X \) is in the ideal. Then \([X, Y] = H, [H, Y] = -2Y\) imply that \( H \) and \( Y \) are also in the ideal which has to be \( \mathfrak{sl}(2, \mathbb{C})\).

The case \( b = 0 \): do the same argument for \([Y, [Y, Z]]\), then the ideal is \( \mathfrak{sl}(2, \mathbb{C}) \) unless \( a = 0 \).

The case \( a = b = 0: [X, X] = 2X, [H, Y] = -2Y \) implies the ideal is \( \mathfrak{sl}(2, \mathbb{C}) \) unless \( c = 0 \). The ideal is \( \{0\} \) if \( a = b = c = 0 \).

**Commutator ideal**: \([g, g] = \text{Span}\{[X, Y]: X, Y \in g\}\).

An ideal is in particular a Lie subalgebra. Keep on taking commutator ideals, get \( g_0 = g, g_{j+1} = [g_j, g] \subset g_j \ldots \) called derived series.

**Solvable**: \( g_j = \{0\} \) for some \( j \).
(If \( g \) is simple, \( g_j = g \) for all \( j \)).

**Levi decomposition**: any Lie algebra is the semi-direct product of a solvable ideal and a semisimple subalgebra.

Similar concept: \( g^0 = g, g^{j+1} = [g, g^j] = \left[ g, [g, \ldots, g] \right] \subset g^j \). Sequence of ideals called **lower central series**.

**Nilpotent**: \( g^j = \{0\} \) for some \( j \).
\( g_j \subset g^j \). Hence nilpotent implies solvable.

For instance, the Lie algebra of nilpotent upper triangular matrices \( n \) is nilpotent (and hence solvable).

Can take the basis \( E_{i,j}, j > i \). Then
\[
[E_{i,j}, E_{k,l}] = 0 \text{ if } j \neq k \text{ and } E_{i,i} \text{ if } j = k.
\]
\( n^k \) is spanned by \( E_{i,j} \) for \( j - i > k \) and hence \( n^{n-1} = \{0\}. \)

The Lie algebra of upper triangular matrices \( u \) is solvable but not nilpotent.

Can take the basis \( E_{i,i}, E_{i,i+1}, E_{i,i+2}, \ldots, E_{1,n} \). Similar as above and \([E_{i,i}, E_{j,j}] = 0\).
\( u_1 = u^1 = n. \) So \( u_n = 0. \) But \( u^i = u^1 \) for all \( i \geq 1. \)

Recall that any Lie group homomorphism \( \Phi: G \to H \) corresponds to a linear map \( \phi: g \to h. \) (In the proof that a continuous homomorphism is smooth.)
Since \( \Phi(gxg^{-1}) = \Phi(g)\Phi(x)\Phi(g)^{-1} \), \( \phi \circ \text{Ad}_g(g) = \text{Ad}_H(\Phi(g)) \circ \phi \).

Then \( \phi \circ \text{ad}_g(X) = \text{ad}_h(\Phi(X)) \circ \phi \), that is \( \phi([X,Y]) = [\Phi(X),\Phi(Y)] \).

Thus Lie group homomorphism corresponds to a Lie algebra homomorphism.

In particular take \( \Phi = \text{Ad} : G \to GL(g) \). It corresponds to the Lie algebra homomorphism \( \phi = \text{ad} : g \to gl(g) \). Hence

\[
\text{Ad}(\exp X) = \exp(\text{ad} X) = \sum_{n=0}^{\infty} \frac{(\text{ad} X)^n}{n!}.
\]

Thus \( \exp X \) commutes with \( \exp tY \) if \( [X,Y] = 0 \) using that \( \exp \circ \text{Ad}(g) = \text{Conj}(g) \circ \exp \) (because conjugation pushes a left invariant vector field to a left invariant one).

Take \( G = GL(n, \mathbb{C}) \). Then

\[
e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{(\text{ad} X)^n \cdot Y}{n!} = \sum_{n=0}^{\infty} \frac{[X,[X,\ldots,[X,Y]]]}{n!}.
\]

If \( G \) is connected, then \( \Phi \) is determined by \( \phi \):

Recall that any element \( g = e^{X_1} \cdot \ldots \cdot e^{X_n} \) if \( G \) is connected. \( \Phi(g) = e^{\Phi(X_1)} \cdot \ldots \cdot e^{\Phi(X_n)} \).

In particular \( g \) commutative implies (connected) \( G \) commutative. (ad=Id, which is the tangent map for both \( Ad \) and \( \text{Id} \).)

Any real Lie algebra (whose underlying vector space is over \( \mathbb{R} \)) can be complexified. For instance, \( \text{gl}(n, \mathbb{R}) \otimes \mathbb{C} \cong \text{gl}(n, \mathbb{C}) \cong \text{u}(n) \otimes \mathbb{C} \).

Thm:

Suppose \( G \) is connected. There exists a one-to-one correspondence between connected normal Lie subgroups and ideal of Lie algebras.

Recall: \( H \) is normal means \( GHG^{-1} = H \).

Lie subgroup means a subgroup which is also an immersed submanifold.

Proof:

Already have correspondence between connected Lie subgroups and Lie subalgebras (Frobenius theorem). (Don't need \( G \) connected for this.)

\( \Rightarrow \) For \( X \in g, Y \in \mathfrak{h} \), \( \exp tX \exp sY \exp -tX \in H \). Taking derivatives gives \([X,Y] \in \mathfrak{h} \).

\( \Leftarrow \) \( \text{Ad}(e^X) \cdot Y = \sum_{n=0}^{\infty} \frac{(\text{ad} X)^n \cdot Y}{n!} \in \mathfrak{h} \).

Since \( \exp \circ \text{Ad}(e^X) = \text{Conj}(e^X) \circ \exp, e^X \cdot e^Y \cdot e^{-X} \in H \).

Since \( H \) is connected, any \( h = e^{Y_1} \cdot \ldots \cdot e^{Y_n} \). Hence \( e^X \cdot h \cdot e^{-X} \in H \).

Since \( G \) is connected, any \( g = e^{X_1} \cdot \ldots \cdot e^{X_n} \). Hence \( g \cdot h \cdot g^{-1} \in H \).

Prop:

If \( G \to H \) induces an isomorphism \( g \to \mathfrak{h} \) and \( H \) is simply connected, then \( G \to H \) is an isomorphism:

\( G \to H \) induces an isomorphism on the tangent spaces together with left multiplications implies \( G \to H \) must...
be a covering map. If $H$ is simply connected, then $G \to H$ must be a diffeomorphism.

**Thm:**
If $G$ is simply connected, then any $g \to \mathfrak{g}$ integrates to $G \to H$.

**Proof:**
Consider the graph of $g \to \mathfrak{g}$ which is a subalgebra in $g \times \mathfrak{g}$. It corresponds to a Lie subgroup $Gr$ in $G \times H$. The first projection $Gr \to G$ is an isomorphism and hence invertible:
the corresponding map $\mathfrak{g} \to \mathfrak{g}$ is an isomorphism, and $G$ is simply connected.
Thus have $G \to Gr$. Compose it with the second projection $Gr \to H$ to get $G \to H$.

**Conclusion:**

<table>
<thead>
<tr>
<th>$X \in \mathfrak{g}$</th>
<th>$all \ g = e^{X_1} \ldots e^{X_n} \in G$</th>
<th>if $G$ is connected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lie subalgebra</td>
<td>$\leftrightarrow$ connected Lie subgroup (which can be immersed)</td>
<td></td>
</tr>
<tr>
<td>ideal</td>
<td>$\leftrightarrow$ connected normal Lie subgroup</td>
<td>if $G$ is connected</td>
</tr>
<tr>
<td>$\exp(\text{ad}(X))$</td>
<td>$\leftrightarrow$ $\text{Ad}(\exp X)$</td>
<td></td>
</tr>
<tr>
<td>$[\mathfrak{g}, \mathfrak{g}] = 0$</td>
<td>$\leftrightarrow$ $\mathfrak{g}h = \mathfrak{g}g$</td>
<td>if $G$ is connected</td>
</tr>
<tr>
<td>group homomorphism</td>
<td>$\leftrightarrow$ algebra homomorphism</td>
<td>if $G$ is simply connected</td>
</tr>
</tbody>
</table>

**Exercises. (Section 3.9)**

4. Give an example of $G \subseteq GL(n, \mathbb{C})$ and $X \in \mathfrak{gl}(n, \mathbb{C})$ such that $e^X \in G$ but $X \notin \text{Lie}(G)$.

21. Let $A \in SL(n, \mathbb{R})$ which has an eigenvalue in $\mathbb{C} - \mathbb{R}$. Show that

$$A = C \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} C^{-1}$$

for some invertible matrix $C$. 
