More on Lie algebras

Wednesday, February 14, 2018 8:27 AM

Simple Lie algebra: $\dim \mathfrak{g} \geq 2$ and the only ideals are $\{0\}$ and itself.
Have classification by Dynkin diagram.

For instance, $\mathfrak{sl}(2, \mathbb{C})$ **is simple**: take the basis
$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
$[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H$.
Suppose $Z = aX + bY + cH$ is in the ideal. $[X, Z] = bH - 2cX$. $[X, [X, Z]] = -2bX$. If $b \neq 0$, then X is in the ideal. Then $[X, Y] = H, [H, Y] = -2Y$ imply that H and Y are also in the ideal which has to be $\mathfrak{sl}(2, \mathbb{C})$.
The case $b = 0$: do the same argument for $[Y, [Y, Z]]$, then the ideal is $\mathfrak{sl}(2, \mathbb{C})$ unless $a = 0$.
The case $a = b = 0$: $[H, X] = 2X, [H, Y] = -2Y$ implies the ideal is $\mathfrak{sl}(2, \mathbb{C})$ unless $c = 0$. The ideal is $\{0\}$ if $a = b = c = 0$.

Commutator ideal: $[\mathfrak{g}, \mathfrak{g}] = \mathrm{Span}\{[X, Y]: X, Y \in \mathfrak{g}\}$.

An ideal is in particular a Lie subalgebra.
Keep on taking commutator ideals, get $\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_{j+1} = [\mathfrak{g}_j, \mathfrak{g}_j] \subset \mathfrak{g}_j, \ldots$ called derived series.
Solvable: $\mathfrak{g}_j = \{0\}$ for some j.
(If $\mathfrak{g}$ is simple, $\mathfrak{g}_j = \mathfrak{g}$ for all j.)

Levi decomposition: any Lie algebra is the semi-direct product of a solvable ideal and a semisimple subalgebra.

Similar concept: $\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^{j+1} = [\mathfrak{g}, \mathfrak{g}^j] = [\mathfrak{g}, [\mathfrak{g}, [\ldots, \mathfrak{g}]]] \subset \mathfrak{g}^j$. Sequence of ideals called **lower central series**.
Nilpotent: $\mathfrak{g}^j = \{0\}$ for some j.
$\mathfrak{g}_j \subset \mathfrak{g}^j$. Hence nilpotent implies solvable.

For instance, the Lie algebra of nilpotent upper triangular matrices $\mathfrak{n}$ is nilpotent (and hence solvable).
Can take the basis $E_{i,j}, j > i$. Then
$[E_{i,j}, E_{k,l}] = 0$ if $j \neq k$ and $= E_{i,l}$ if $j = k$.
$\mathfrak{n}^k$ is spanned by $E_{i,j}$ for $j - i > k$ and hence $\mathfrak{n}^{n-1} = \{0\}$.

The Lie algebra of upper triangular matrices $\mathfrak{u}$ is solvable but not nilpotent.
Can take the basis $E_{i,i}, E_{i,i+1}, E_{i,i+2}, \ldots, E_{1,n}$. Similar as above and $[E_{i,i}, E_{j,j}] = 0$.
$\mathfrak{u}_1 = \mathfrak{u}^1 = \mathfrak{n}$. So $\mathfrak{u}_n = 0$. But $\mathfrak{u}^i = \mathfrak{u}^1$ for all $i \geq 1$.

Recall that any Lie group homomorphism $\Phi: G \to H$ corresponds to a linear map $\phi: \mathfrak{g} \to \mathfrak{h}$. (In the proof that a continuous homomorphism is smooth.)

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\phi"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{h} \arrow[d, "\exp"] \\
G \arrow[r, "\Phi"'] & H
\end{tikzcd}$$

Since $\Phi(gxg^{-1}) = \Phi(g)\Phi(x)\Phi(g)^{-1}$, $\phi \circ \mathrm{Ad}_{\mathrm{G}}(g) = \mathrm{Ad}_{\mathrm{H}}(\Phi(g)) \circ \phi$.
Then $\phi \circ \mathrm{ad}_{\mathfrak{g}}(X) = \mathrm{ad}_{\mathfrak{h}}(\phi(X)) \circ \phi$, that is $\phi([X,Y]) = [\phi(X), \phi(Y)]$.
Thus **Lie group homomorphism corresponds to a Lie algebra homomorphism**.

In particular take $\Phi = \mathrm{Ad}\colon\ G \to GL(\mathfrak{g})$. It corresponds to the Lie algebra homomorphism $\phi = \mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.
Hence

$$\mathbf{Ad}(\mathbf{exp}\,\boldsymbol{X}) = \mathbf{exp}(\mathbf{ad}\,\boldsymbol{X}) = \sum_{n=0}^{\infty} \frac{(\mathrm{ad\ X})^n}{n!}.$$

Thus **exp $\boldsymbol{X}$ commutes with exp $\boldsymbol{tY}$ if $[\boldsymbol{X}, \boldsymbol{Y}] = \mathbf{0}$** using that $\exp \circ \mathrm{Ad(g)} = \mathrm{Conj(g)} \circ \exp$ (because conjugation pushes a left invariant vector field to a left invariant one).

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "Ad(g)"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "Conj(g)"'] & G \end{tikzcd}$$

Take $G = GL(n, \mathbb{C})$. Then

$$\boldsymbol{e^X\, Y\, e^{-X}} = \sum_{\boldsymbol{n=0}}^{\infty} \frac{\boldsymbol{(\mathrm{ad\ X})^n \cdot Y}}{\boldsymbol{n!}} = \sum_{\boldsymbol{n=0}}^{\infty} \frac{\big[\boldsymbol{X}, [\boldsymbol{X}, \dots, [\boldsymbol{X}, \boldsymbol{Y}]]\big]}{\boldsymbol{n!}}.$$

If G is connected, then $\boldsymbol{\Phi}$ is determined by $\boldsymbol{\phi}$:
Recall that any element $g = e^{X_1} \cdot \ldots \cdot e^{X_n}$ if G is connected. $\Phi(g) = e^{\phi(X_1)} \cdot \ldots \cdot e^{\phi(X_n)}$.
In particular **$\mathfrak{g}$ commutative implies (connected) G commutative**. (ad=Id, which is the tangent map for both Ad and Id.)

Any real Lie algebra (whose underlying vector space is over $\mathbb{R}$) can be complexified. For instance, $\mathfrak{gl}(n, \mathbb{R}) \otimes \mathbb{C} \cong \mathfrak{gl}(n, \mathbb{C}) \cong \mathfrak{u}(n) \otimes \mathbb{C}$.

Thm:
Suppose G is connected. There exists a one-to-one correspondence between connected normal Lie subgroups and ideal of Lie algebras.

Recall: H is normal means $GHG^{-1} = H$.
Lie subgroup means a subgroup which is also an immersed submanifold.

Proof:
Already have correspondence between connected Lie subgroups and Lie subalgebras (Frobenius theorem). (Don't need G connected for this.)

=>) For $X \in \mathfrak{g}, Y \in \mathfrak{h}$, $\exp tX \exp sY \exp -tX \in H$. Taking derivatives gives $[X, Y] \in \mathfrak{h}$.
<=) $Ad(e^X) \cdot Y = \sum_{n=0}^{\infty} \frac{(\mathrm{ad\ X})^n \cdot Y}{n!} \in \mathfrak{h}$.
Since $\exp \circ \mathrm{Ad}(e^X) = \mathrm{Conj}(e^X) \circ \exp$, $e^X \cdot e^Y \cdot e^{-X} \in H$.
Since H is connected, any $h = e^{Y_1} \cdot \ldots \cdot e^{Y_n}$. Hence $e^X \cdot h \cdot e^{-X} \in H$.
Since G is connected, any $g = e^{X_1} \cdot \ldots \cdot e^{X_n}$. Hence $g \cdot h \cdot g^{-1} \in H$.

Prop:
If $\boldsymbol{G \to H}$ induces an isomorphism $\mathfrak{g} \to \mathfrak{h}$ and H is simply connected, then $\boldsymbol{G \to H}$ is an isomorphism:
$G \to H$ induces an isomorphism on the tangent spaces together with left multiplications implies $G \to H$ must

be a covering map. If H is simply connected, then $G \to H$ must be a diffeomorphism.

Thm:
If G is simply connected, then any $\mathfrak{g} \to \mathfrak{h}$ integrates to $G \to H$.

Proof:
Consider the graph of $\mathfrak{g} \to \mathfrak{h}$ which is a subalgebra in $\mathfrak{g} \times \mathfrak{h}$. It corresponds to a Lie subgroup Gr in $G \times H$.
The first projection $Gr \to G$ is an isomorphism and hence invertible:
the corresponding map $\mathfrak{gr} \to \mathfrak{g}$ is an isomorphism, and G is simply connected.
Thus have $G \to Gr$. Compose it with the second projection $Gr \to H$ to get $G \to H$.

Conclusion:

$X \in \mathfrak{g}$		all $g = e^{X_1} \ldots e^{X_n} \in G$	if G is connected
Lie subalgebra	<->	connected Lie subgroup (which can be immersed)	
ideal	<->	connected normal Lie subgroup	if G is connected
$\exp(\mathrm{ad}(X))$	<->	$\mathrm{Ad}(\exp X)$	
$[\mathfrak{g}, \mathfrak{g}] = 0$	<->	gh=hg	if G is connected
group homomorphism	<->	algebra homomorphism	if G is simply connected

Exercises. (Section 3.9)

4. Give an example of $G \subset GL(n, \mathbb{C})$ and $X \in \mathfrak{gl}(n, \mathbb{C})$ such that $e^X \in G$ but $X \notin \mathrm{Lie}(G)$.
21. Let $A \in SL(n, \mathbb{R})$ which has an eigenvalue in $\mathbb{C} - \mathbb{R}$. Show that
$$A = C \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} C^{-1}$$
for some invertible matrix C.