Complex representation: $\Pi: G \to GL(V)$ where V is a complex vector space.
For Lie algebras, $\pi: \mathfrak{g} \to \mathfrak{gl}(V)$.
Recall group homomorphism induces algebra homomorphism (and vice versa if G is simply connected).
Thus $\boldsymbol{\Pi}$ **gives** $\boldsymbol{\pi} = \left.\frac{\boldsymbol{d}}{\boldsymbol{dt}}\right|_{\boldsymbol{t=0}} \boldsymbol{\Pi(e^{tx})}$. Also $\pi \circ \mathrm{Ad}_g = \mathrm{Ad}_{\Pi(g)} \circ \pi = \Pi(g) \cdot \pi \cdot \Pi(g)^{-1}$.

Examples: standard representations; adjoint representations.

Faithful: the homomorphism is injective.
Irreducible: V has no non-trivial invariant subspace.
Unitary: there is a Hermitian metric on V such that $\Pi: G \to U(V)$.
Morphism between representation: $V \to W$ which commutes with the action (intertwining).

Suppose G is connected.

1. $\boldsymbol{\Pi}$ is irreducible if and only if $\boldsymbol{\pi}$ is:
$W \subset V$ invariant under G => invariant under $\mathfrak{g}$.
If W is invariant under $\mathfrak{g}$, then it is invariant under
$\exp tX \in G$. But any element in G can be written as product of these.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\pi"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{gl}(V) \arrow[d, "\exp"] \\ G \arrow[r, "\Pi"'] & GL(V) \end{tikzcd}$$

2. $\boldsymbol{\Pi_1 \cong \Pi_2}$ if and only if $\boldsymbol{\pi_1 \cong \pi_2}$:
=> is obvious since $V \to W$ intertwines with the group actions implies it intertwines with the algebra actions.
<=) $V \to W$ intertwines with $\pi_i(X)$, and hence $\exp \pi_i(X) = \Pi(\exp X)$, and hence $\Pi(g)$ for arbitrary g.

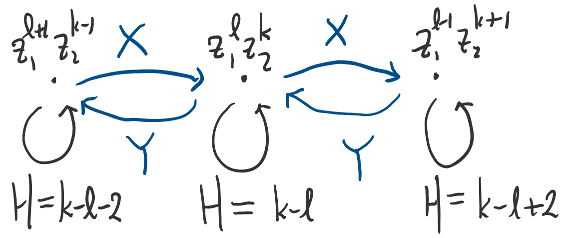
3. Given $\boldsymbol{\Pi}$. $\boldsymbol{\Pi: G \to U(V)}$ <-> $\boldsymbol{\pi: \mathfrak{g} \to \mathfrak{u}(V)}$:
-> we already have $\pi: \mathfrak{g} \to \mathfrak{gl}(V)$. It is obvious that the image is contained in $\mathfrak{u}(V)$.
<- $\Pi(\exp X) = \exp(\pi(X)) \in U(V)$. Any $g \in G$ is a product of $\exp X$.

Most important example:
$SU(2) \to GL(\mathbb{C}^2)$, and hence $SU(2) \to SU\left(Sym^m\left(\left(\mathbb{C}^2\right)^*\right)\right)$. Explicitly
$g \cdot f = (g^{-1})^* f = f(g^{-1} \cdot z)$.
Induces $\mathfrak{su}(2) \to \mathfrak{su}\left(Sym^m\left(\left(\mathbb{C}^2\right)^*\right)\right)$.

$$X \cdot f = \left.\frac{d}{dt}\right|_{t=0} f\left(g_t^{-1} \cdot z\right) = df\Big|_z \cdot (-X \cdot z).$$

Complexify: $\mathfrak{su} \otimes \mathbb{C} = \mathfrak{sl}_{\mathbb{C}}$ (any complex matrix is a sum of Hermitian and skew-Hermitian matrices.)
Have $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}\left(Sym^m\left(\left(\mathbb{C}^2\right)^*\right)\right)$ defined by the same formula.
Suppose $f = z_1^l z_2^k$ $(l + k = m)$. $df = l z_1^{l-1} z_2^k \, dz_1 + k z_1^l z_2^{k-1} dz_2$.
Let $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
$X \cdot z_1^l z_2^k = l z_1^{l-1} z_2^k \, (-z_2) = -l z_1^{l-1} z_2^{k+1}$.
$Y \cdot z_1^l z_2^k = k z_1^l z_2^{k-1} \, (-z_1) = -k z_1^{l+1} z_2^{k-1}$.
$H \cdot z_1^l z_2^k = l z_1^{l-1} z_2^k \, (-z_1) + k z_1^l z_2^{k-1} (z_2) = (-l + k) z_1^l z_2^k$.
Eigenspace decomposition of H:

$Sym^m\left(\left(\mathbb{C}^2\right)^*\right) = \bigoplus_{l+k=m} \mathbb{C}\cdot z_1^l z_2^k$ with eigenvalues $k-l$.



$Sym^m\left(\left(\mathbb{C}^2\right)^*\right)$ is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$:

Suppose W is an invariant subspace and $0 \neq f \in W$. Apply X enough times, it becomes zero.

$0 \neq X^q \cdot f \in \mathbb{C}\cdot z_2^m$ for some q. Hence W contains $\mathbb{C}\cdot z_2^m$. Now take Y^p, then W contains everything.

Exercises. (Section 4.9)

2. Show that the adjoint representation and the standard representation of $\mathfrak{so}(3)$ are isomorphic.

13. Let π be a representation of $\mathfrak{sl}(2,\mathbb{C})$. Show that the eigenvalues of $\pi(H)$ for $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are integers (by using $e^{2\pi i H} = Id \in SU(2)$).