# Schur's Lemma

Monday, February 26, 2018 8:34 AM

**Completely reducible representation**: direct sum of irreducible ones.
**Completely reducible Lie group/algebra:** every representation is completely reducible.

**Non-example**: $\mathbb{R} \to \mathrm{GL}(2, \mathbb{C}), x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.
Preserve $e_1$, but there is no other invariant subspace.

**If $V$ is completely reducible, then given any invariant $U \subset V$, there is $U'$ such that $U \oplus U' = V$**:
$V = V_1 \oplus \cdots \oplus V_k$. There is $V_i \not\subset U$ (otherwise $U = V$), say $i = 1$.
$V_i$ irreducible implies $V_1 \cap U = \{0\}$. $U \subset V_2 \oplus \cdots \oplus V_k$.
Keep doing this until $U = V_j \oplus \cdots \oplus V_k$. Take $U' = V_1 \oplus \cdots \oplus V_{j-1}$.
By this proof,
**every invariant subspace U of a completely reducible V is completely reducible.**

**If $\Pi$ or $\pi$ is unitary, then it is completely reducible:**
Take invariant subspaces and its orthogonal complement.

**If G is compact, then it is completely reducible:**
Any representation is unitary with respect to a metric. Take arbitrary metric h, and take the average to make it G-invariant:

$$\langle v, w \rangle_G \coloneqq \int_G \langle \Pi(\mathrm{g}) \cdot v, \Pi(\mathrm{g}) \cdot w \rangle \ \nu_G$$

where $\nu_G$ is a non-zero right-G-invariant top form on G. It is G-invariant:

$$\langle \Pi(h) \cdot v, \Pi(h) \cdot w \rangle_G \coloneqq \int_G \langle \Pi(\mathrm{g}) \cdot \Pi(h) \cdot v, \Pi(\mathrm{g}) \cdot \Pi(h) \cdot w \rangle \ \nu_G$$

$$= \int_G \langle \Pi(\mathrm{R_h} \cdot \mathrm{g}) \cdot v, \Pi(\mathrm{R_h} \cdot \mathrm{g}) \cdot w \rangle \ \nu_G$$

$$= \int_G \langle \Pi(\mathrm{g}) \cdot v, \Pi(\mathrm{g}) \cdot w \rangle \ (R_h^*)^{-1} \cdot \nu_G = \int_G \langle \Pi(\mathrm{g}) \cdot v, \Pi(\mathrm{g}) \cdot w \rangle \ \nu_G = \langle v, w \rangle_G.$$

Hence $\mathfrak{sl}(\mathrm{n}, \mathbb{C}) = \mathfrak{su}(n) \otimes \mathbb{C}$ is completely reducible.

**Schur's Lemma:**
If V, W are irreducible representations, then any non-zero morphism $\phi: V \to W$ is

an isomorphism. Any $\phi_1$ and $\phi_2 \neq 0$ are related by $\phi_1 = \lambda\phi_2$.
If V=W, then $\phi = \lambda \cdot Id$.
**Proof:**
Consider Ker $\phi$, which is a representation since $\phi$ intertwines with G-action. V is irreducible and $\phi \neq 0$ imply Ker $\phi = 0$.
Consider Im $\phi$. W is irreducible and and $\phi \neq 0$ imply Im $\phi = W$.

If V=W, consider eigenvalues of $\phi$. There exists an eigenvalue $\lambda$ since we work over $\mathbb{C}$. The eigenspace is G-invariant since $\phi$ is intertwining. V is irreducible implies the eigenspace is V.

For $\phi_1, \phi_2 : V \to W$, consider $\phi_2^{-1} \circ \phi_1 : V \to V$, which is $\lambda \cdot Id$.

**Corollary:** for a commutative Lie group/algebra, irreducible representation is one-dimensional.
**Proof:** $\Pi(g): V \to V$ for any g is a morphism: $\Pi(h) \circ \Pi(g) = \Pi(g) \circ \Pi(h)$.
Schur's lemma gives $\Pi(g) = \lambda_g \cdot \mathrm{Id}$. Any subspace is invariant. Irreducible => 1d.

Same proof gives
**Corollary:** $\Pi(g) = \lambda_g \cdot \mathrm{Id}$ if g belongs to center and $\Pi$ is irreducible.

**Heisenberg group**

$$H = \left\{ \begin{pmatrix} 1 & x & \theta \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

which is universal cover of
$G = \mathbb{R} \times \mathbb{R} \times \mathbb{S}^1$ with
$(x_1, y_1, u_1) \cdot (x_2, y_2, u_2) = (x_1 + x_2, y_1 + y_2, e^{2\pi i\, x_1 y_2}\, u_1\, u_2)$.
$u = e^{2\pi i \theta}$. (Inverse of $(x, y, u)$ is $(-x, -y, e^{2\pi i\, xy} u^{-1})$.)

$$\boldsymbol{G} = \boldsymbol{H}/\boldsymbol{N} \text{ where } N = \left\{ \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\}.$$

$$(x_1, y_1, u_1) \cdot (x_2, y_2, u_2) \cdot \left(-x_1, -y_1, e^{2\pi i x_1 y_1} u_1^{-1}\right)$$
$$= (x_2, y_2, e^{-2\pi i (x_1 + x_2) y_1} e^{2\pi i\, x_1\, y_2} e^{2\pi i x_1 y_1} u_2)$$
$$= (x_2, y_2, e^{2\pi i (x_1 y_2 - x_2 y_1)} u_2).$$

Thus **center is $\boldsymbol{x} = \boldsymbol{y} = \mathbf{0}$**.

**Theorem:** G has no faithful representation.
**Proof:** Lift to representation $\widetilde{\Pi}$ of $H$ with Ker $\supset N$.

Claim: $\mathrm{Ker}(\widetilde{\Pi}) \supset Z(H)$, and hence $\mathrm{Ker}(\Pi) \supset Z(G) = \{0\} \times \{0\} \times \mathbb{S}^1$, not faithful.

Proof of claim: consider the Lie algebras. Want to see $\mathrm{Ker}(\pi) \ni \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = Z$.

Then $\mathrm{Ker}\left(\widetilde{\Pi}\right) \ni e^{tZ}$.

$e^Z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in N$. Thus $e^{\pi(Z)} = \widetilde{\Pi}\left(e^Z\right) = \mathrm{Id}$. Hence $e^{k\,\pi(Z)} = \mathrm{Id}$ for all $k \in \mathbb{Z}$.

But $e^{k\,\pi(Z)}$ is polynomial in k since $\pi(Z)$ is nilpotent. Then
the equation has infinitely many roots implies the polynomial is identically zero.
$\pi(Z)$ is nilpotent: need to show it only has zero eigenvalue. Note that $Z = [X, Y]$.
Hence $\mathrm{tr}(\pi(Z)|_U) = 0$ for any Z-eigenspace U (which is perserved by $\pi(X), \pi(Y)$ since they commute with $\pi(Z)$). Thus the eigenvalue must be zero.

**Exercises. (Section 4.9)**

5. Consider the standard representation of the Heisenberg group acting on $\mathbb{C}^3$. Determine all the invariant subspaces. Is the standard representation completely reducible?
11. Let $V$ be an irreducible representation over $\mathbb{C}$. Show that every non-trivial invariant subspace of $V \oplus V$ is of the form $\{(\lambda_1 v, \lambda_2 v) : v \in V\} \cong V$ for some $(\lambda_1, \lambda_2) \in \mathbb{C}^2 - \{0\}$.