Representation of sl(3,C)

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Basis:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$Y_i = X_i^T$.

Then $[H_1, H_2] = 0, [H_1, X_1] = 2X_1, [H_1, X_2] = -X_2, [H_1, X_3] = X_3, [H_2, X_1] = -X_1, [H_2, X_3] = X_3, [H_1, Y_1] = -2Y_1$, and so on;

$[X_1, Y_1] = H_1, [X_2, Y_2] = H_2, [X_3, Y_3] = H_1 + H_2, [X_1, X_2] = X_3, [Y_1, Y_2] = -Y_3$ and so on.

Given a representation π, consider simultaneous eigenspaces of $\pi(H_i)$: first take an eigenspace U of $\pi(H_1)$; since $[\pi(H_1), \pi(H_2)] = 0$, $\pi(H_2)$ preserves U and has an eigenspace in U.

Given a simultaneous eigenvector v, have eigenvalues m_i of $\pi(H_i)$.

$\mu = (m_1, m_2) \in \mathfrak{h}^*$ is called a **weight** for π, where $\mathfrak{h} = \text{Span}(H_1, H_2)$.

The space of all such simultaneous eigenvectors is called the μ-**weight space**. Its dimension is called to be the multiplicity of μ.

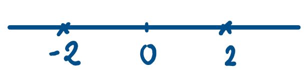
m_i are integers:

restrict the representation to $\langle H_i, X_i, Y_i \rangle \cong \mathfrak{sl}(2, \mathbb{C})$.

Now apply the above concept to the adjoint representation. The corresponding non-zero weights are called **roots**; elements in a weight space are called **root vectors**.

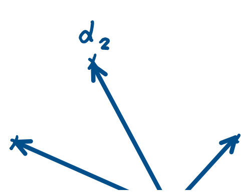
For $\mathfrak{sl}(2, \mathbb{C})$ the root vectors and roots are

X	2
Y	-2

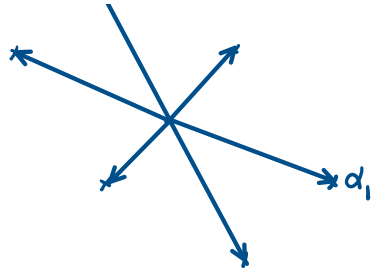


For $\mathfrak{sl}(3, \mathbb{C})$ the root vectors and roots are

X_1	$\alpha_1 = (2, -1)$
X_2	$\alpha_2 = (-1, 2)$
X_3	$(1,1)$
Y_1	$(-2,1)$



X_3	(1,1)
Y_1	(-2,1)
Y_2	(1,-2)
Y_3	(-1,-1)



α_i are called positive simple roots. All other roots are linear combinations of them with coefficients either all non-negative or all non-positive.

Prop: Let Z_α be a root vector.
$\pi(Z_\alpha)$ sends μ-weight space to $(\mu+\alpha)$-weight space.
Proof: let $0 \neq v \in \mu = (m_1, m_2)$-weight space of π. $\alpha = (a_1, a_2)$.
$\pi(H_i)\cdot\pi(Z_\alpha)v = \pi(Z_\alpha)\cdot\pi(H_i)v + \pi([H_i, Z_\alpha])v = m_i\,\pi(Z_\alpha)v + a_i\,\pi(Z_\alpha)v$
$= (m_i + a_i)\pi(Z_\alpha)v.$
QED

Def: The weights μ_1 is **higher** than μ_2 if $\mu_1 - \mu_2 = a\alpha_1 + b\alpha_2$ for some $a, b \in \mathbb{R}_+$.
It gives a partial ordering. (α_1 is neither higher nor lower than α_2.)

Highest weight representation with weight μ:
There exists a weight vector $v \neq 0$ corresponding to μ such that $\pi(X_j)\cdot v = 0$ for all j, and v is cyclic (that is $V = \mathfrak{g}\cdot v$).

By definition μ is really the highest weight and it has multiplicity one:
By keep on taking Y_i on v, get an invariant subspace which must be V. (**it is invariant**: A product of operations can always be expressed in terms of
$\pi(Y_1)^{p_1}\pi(Y_2)^{p_2}\pi(Y_3)^{p_3}\pi(H_1)^{q_1}\pi(H_2)^{q_2}\pi(H_3)^{q_3}\pi(X_1)^{r_1}\pi(X_2)^{r_2}\pi(X_3)^{r_3}$.
Acting on v, it becomes scaling of $\pi(Y_1)^{p_1}\pi(Y_2)^{p_2}\pi(Y_3)^{p_3}$.)
Y_i decrease the weight. Hence μ is the unique highest weight, and the μ-weight space is one-dimensional: $\mathbb{C}\cdot v$.

CAUTION*:* V is cyclic does not imply it is irreducible:
For instance take $V_2 \oplus V_3$ of $\mathfrak{sl}(2,\mathbb{C})$. Then $v_2 + v_3$ is cyclic (where v_i are highest weight vector of V_i).
Indeed irreducible <=> every non-zero vector is cyclic.

Irreducible <=> highest weight representation.
Proof.

=>)
Irreducible V is a direct sum of weight spaces:
There exists a μ-weight space V_μ over $\mathbb{C}$. Z_α sends V_μ to $V_{\mu+\alpha}$ (and H_i preserve V_μ). Then keep on taking Z_α, get an invariant subspace which is V itself. $V_{\mu_1} \cap V_{\mu_2} = \{0\}$ if $\mu_1 \neq \mu_2$.

Since V is finite-dimensional, there must be a highest weight. A corresponding weight vector v must have $\pi(X_j) \cdot v = 0$. v is cyclic since V is irreducible. Hence V is a highest weight representation.

<=)
Any finite dimensional representation of $\mathfrak{sl}(3, \mathbb{C})$ corresponds to that of SU(3) which is simply connected and compact. Hence it must be completely reducible. Each irreducible part is a direct sum of weight spaces. Hence the highest weight space, which has dimension one, must belong to one irreducible part. But it is cyclic, and hence the whole V is that part.
QED.

Theorem:
Irreducible representation V of $\mathfrak{sl}(3, \mathbb{C})$ <-> $(m_1, m_2) \in \mathbb{Z}^2_{\geq 0}$
where the correspondence is given by taking the highest weight.

Proof:
->
Take the highest weight.
m_i are non-negative:
Restrict to $\langle H_i, X_i, Y_i \rangle \cong \mathfrak{sl}(2, \mathbb{C})$.

This is injective:
Suppose V and W have the same highest weight with weight vectors v and w. Consider the subspace U generated by $(v, w) \in V \oplus W$. (v, w) is a weight vector (since v and w have the same weight) which is highest cyclic. Hence U is irreducible. The projection maps $U \to V$ and $U \to W$ are morphisms and non-zero, and hence are isomorphisms by Schur's Lemma.

This is surjective:
Standard representation $V = \mathbb{C}^3$: since

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

the standard basic vectors are weight vectors with $\mu = (1,0), (-1,1), (0,-1)$.
Recall $\alpha_1 = (2,-1), \alpha_2 = (-1,2)$. Hence $(1,0)$ is the highest weight and e_1 is a highest weight vector which is cyclic.

Dual of standard representation V^*. The action is right multiplication by g^{-1} on row vectors. The standard row vectors have weights $\mu = (-1,0), (1,-1), (0,1)$. $(0,1)$ is the highest weight and e_3^* is a highest weight vector.

Then consider $\boldsymbol{V^{\otimes m_1} \otimes (V^*)^{\otimes m_2}}$.
$v_{m_1,m_2} = e_1^{\otimes m_1} \otimes (e_3^*)^{\otimes m_2}$ has weight (m_1, m_2).
Take the invariant subspace generated by v_{m_1,m_2}. Then it is a highest weight representation with highest weight (m_1, m_2). Hence it is irreducible.
QED

Exercises. (Section 6.9)

6. Find the weights and multiplicities of the (2,0)-highest weight representation of $\mathfrak{sl}(3,\mathbb{C})$.
8. Show that the space of homogeneous polynomials of degree m in three variables is the (0,m)-highest weight representation of $\mathfrak{sl}(3,\mathbb{C})$.