# Weyl group of SU(3)

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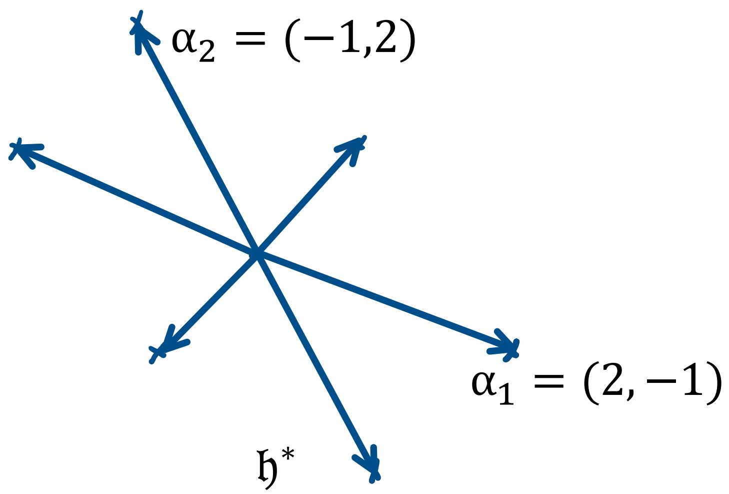
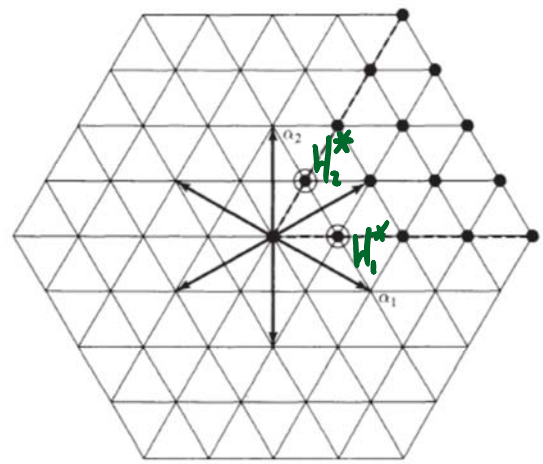
$\mathfrak{h}_{\mathbb{C}} \coloneqq \mathbf{Span}_{\mathbb{C}}\{\boldsymbol{H_1}, \boldsymbol{H_2}\} \subset \mathfrak{sl}(3,\mathbb{C}) = \mathfrak{su}(3) \otimes \mathbb{C}$ Cartan subalgebra.
Can identify $\mathfrak{h}_{\mathbb{C}}$ with $\mathfrak{h}_{\mathbb{C}}^*$ via the Killing form $tr(X_1 X_2^*)$.
Then the roots and weights are elements in $\mathfrak{h} \cong \mathfrak{h}^*$.
Can restrict to $\mathfrak{h}_{\mathbb{R}} \cong \mathfrak{h}_{\mathbb{R}}^* \subset i\,\mathfrak{su}(3)$ (where the Killing form is positive).




$e_1^* = (1,0) \in \mathfrak{h}^*$ is identified with $\mathrm{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$,
$e_2^* = (0,1) \in \mathfrak{h}^*$ is identified with $\mathrm{diag}\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$.
Then $\alpha_1 \leftrightarrow \mathrm{diag}(1,-1,0) = H_1$, $\alpha_2 \leftrightarrow \mathrm{diag}(0,1,-1) = H_2$.
They have length $\sqrt{2}$ and $\langle \alpha_1, \alpha_2 \rangle = -1$.
Also $\langle \alpha_i, e_j^* \rangle = \delta_{ij}$.

SU(3) has adjoint action on $\mathfrak{su}(3)$, $i\,\mathfrak{su}(3)$, and hence $\mathfrak{sl}(3,\mathbb{C})$.

N: subgroup of SU(3) preserving $\mathfrak{h}$.
Z: subgroup of SU(3) that fixes every element of $\mathfrak{h}$.
Z is a normal subgroup of N.
$W \coloneqq N/Z$: the Weyl group for SU(3).

**Theorem: For any representation, the set of weights has W-symmetry**. Namely, if $\lambda \in \mathfrak{h}^*$ is a weight, then $w \cdot \lambda$ is also a weight with the same multiplicity.

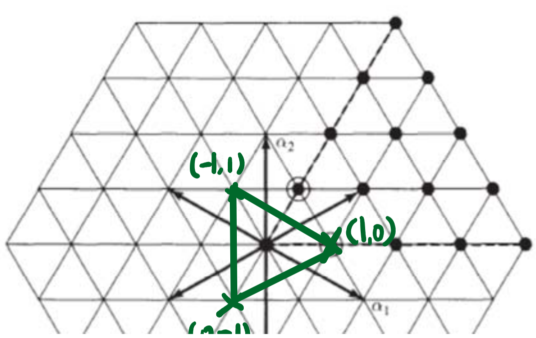
**Proof:**
Consider the λ-weight space $V_\lambda$. Then $U \cdot V_\lambda$ is the $(w \cdot \lambda)$-weight space (where $w = [U]$):
$H \cdot (U \cdot v) = U \cdot (\mathrm{Ad}_{U^{-1}} H) \cdot v = (\lambda, \mathrm{Ad}_{U^{-1}} H)\, U \cdot v$
$= (\mathrm{Ad}_U \lambda, H)\, U \cdot v = (w \cdot \lambda, H)\,(U \cdot v)$.
QED

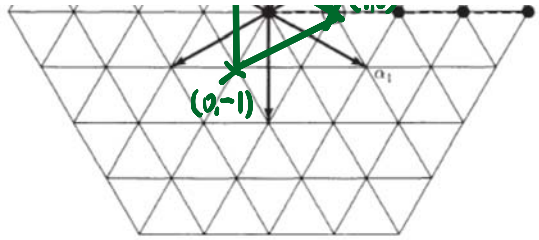
**Prop: W is the permutation group on three elements (namely the three weights of the standard representation).**

**Proof:**
Recall the standard representation $\mathbb{C}^3$ of SU(3) has weight spaces $\mathbb{C} \cdot e_i$ for $i = 1,2,3$. W permutes the weight spaces. This gives $W \to S_3$.

**Surjective**:
Given any permutation, we can realize it by a permutation matrix U which permutes the basic vectors. (It lies in SU(3) since it preserves metric. Take a basic vector $e$ to $-e$ if necessary to keep orientation).
$[U] \in W$: if H is diagonal, $UHU^{-1}$ is still diagonal since $U^{-1}e_i$ is a basic vector and hence H acts by scaling. Thus U preserves $\mathfrak{h}$.

**Injective:**
If $[U] \mapsto \mathrm{Id}$, then U is diagonal. Then $UHU^{-1} = H$ for any $H \in \mathfrak{h}$, and hence $[U] = 1 \in W$.
QED

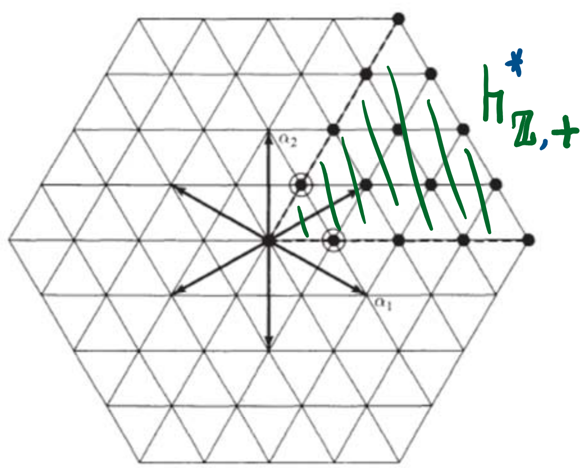





standard representation

**Conclusion: $W \cong S_3$ is the symmetry group of the equilateral triangle, which acts by reflections about $\alpha_i^{\perp}$.**

**Integral structure**:
$\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z} \cdot \{H_1, H_2\} \subset \mathfrak{h}_{\mathbb{R}}$. $\mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z}) \subset \mathfrak{h}_{\mathbb{R}}^*$.
**Dominant structure**:
$\mathfrak{h}_{\mathbb{Z},+} = \mathbb{Z}_{\geq 0} \cdot \{H_1, H_2\}$. $\mathfrak{h}_{\mathbb{Z},+}^* = \mathrm{Hom}(\mathfrak{h}_{\mathbb{Z},+}, \mathbb{Z}_{\geq 0})$.




**Theorem**:
For the highest weight representation $V_\mu$,
$\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ is a weight (with non-zero multiplicity) if and only if $\lambda \in (\mu + \mathbb{Z} \cdot \langle \alpha_1, \alpha_2 \rangle) \cap \mathrm{Conv}(W \cdot \mu)$.

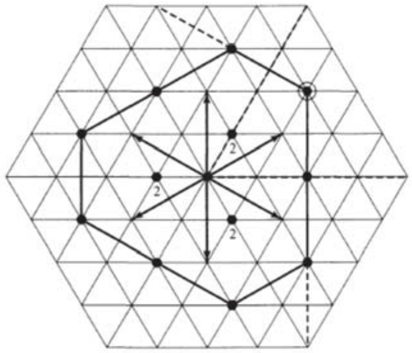


Fig. 5.4. Highest weight (1,2)

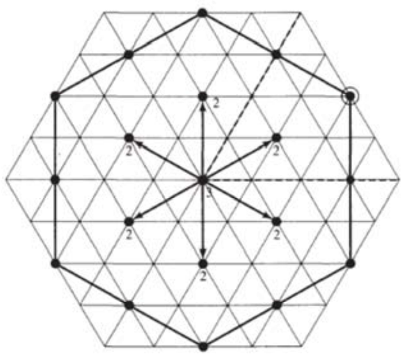
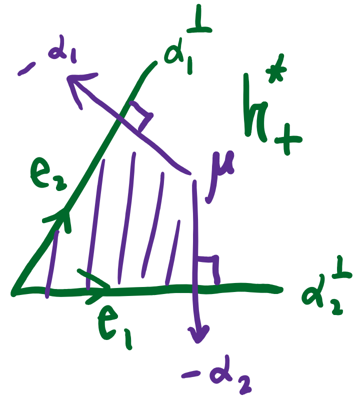


Fig. 5.5. Highest weight (2,2)

Proof:

=>)

$V_\mu$ is spanned by $Y_{k_1} \dots Y_{k_j} \cdot v$ where $v$ is the highest weight vector.
These are weight vectors with weights $\lambda \in \mu - \mathbb{Z} \cdot \langle \alpha_1, \alpha_2 \rangle$.
$w \cdot \lambda \in \mathfrak{h}^*_{\mathbb{Z},+}$ for some $w \in W$. Also $w \cdot \lambda \leq \mu$ since it is still a weight.
Hence it belongs to the shaded quadrilateral.
$w_{\alpha_i}$ acts as reflection about $\alpha_i^\perp$. Hence the vertices of the quadrilateral are $0, \mu, \frac{w_{\alpha_1} \cdot \mu + \mu}{2}, \frac{w_2 \cdot \mu + \mu}{2}$
which all belong to $\mathrm{Conv}(W \cdot \mu)$.




<=) (no hole in between)

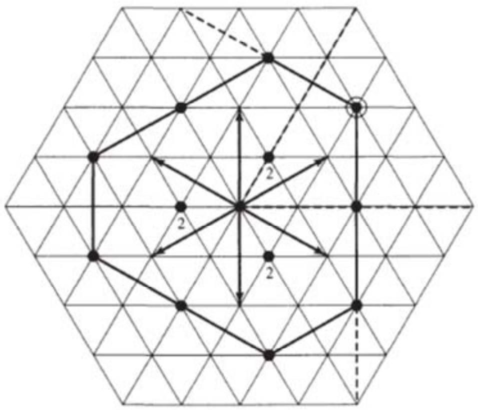



**Fig. 5.4.** Highest weight (1,2)

First each element $w$ in $(\mu - \mathbb{Z}_{\geq 0} \cdot \alpha_i) \cap \mathrm{Conv}(W \cdot \mu)$ (for i=1,2) is a weight by restricting to $\langle H_i, X_i, Y_i \rangle \cong \mathfrak{sl}(2, \mathbb{C})$.
Then each element in $\left(w - \mathbb{Z}_{\geq 0} \cdot \alpha_j\right) \cap \mathrm{Conv}(W \cdot \mu)$ (for $j = 1,2,3$) is a weight by restrictly to $\left\langle H_j, X_j, Y_j \right\rangle \cong \mathfrak{sl}(2, \mathbb{C})$ (where $H_3 = H_1 + H_2$).
This covers all the elements in $(\mu + \mathbb{Z} \cdot \langle \alpha_1, \alpha_2 \rangle) \cap \mathrm{Conv}(W \cdot \mu)$.

# Exercises. (Section 6.9)

10. Classify the irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$ whose sets of weights are invariant under $-\mathrm{Id}$ on $\mathfrak{h}_{\mathbb{R}}^*$.
11. For the highest weight representation $V_\lambda$, show that the $(\mu - \alpha_1 - \alpha_2)$-weight space has multiplicity at most two, and it is spanned by
    $Y_1 \cdot Y_2 \cdot v_0$ and $Y_2 \cdot Y_1 \cdot v_0$
    where $v_0$ is a highest weight vector.