Semi-simple Lie algebra

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Reductive: $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ where $\mathfrak{k}$ is the Lie algebra of a compact Lie group K. ($\mathfrak{k}$ is called a compact real form.)
Semi-simple: reductive and trivial center.

ex. $\mathfrak{sl}(n, \mathbb{C})$ is semi-simple. $\mathfrak{gl}(n, \mathbb{C})$ is reductive but not semi-simple.

We have a K-invariant **Hermitian** metric on $\mathfrak{k}_{\mathbb{C}}$ which restricts to usual metric on $\mathfrak{k}$.
(This is NOT the Killing form since it is Hermitian instead of bilinear.)
This makes the adjoint representation $\mathfrak{k}$ unitary: $\langle \mathrm{ad}_X(Y), Z\rangle = -\langle Y, \mathrm{ad}_X(Z)\rangle$.
Then for the complexified action by $X \in \mathfrak{k}_{\mathbb{C}}$,
$\langle \mathrm{ad}_X(Y), Z\rangle = -\langle Y, \mathrm{ad}_{\bar{X}}(Z)\rangle$.

Prop. Reductive $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{z}$ where $\mathfrak{g}_1$ is semisimple and $\mathfrak{z}$ is the center.
Proof:
$\mathfrak{z}$ is an ideal, and so is $\mathfrak{g}_1 \coloneqq \mathfrak{z}^{\perp}$.
$\mathfrak{g}_1$ is semi-simple: it has trivial center since any central element of $\mathfrak{g}_1$ would be central element of $\mathfrak{g}$ and hence belongs to $\mathfrak{z}$.
Compact real form: take away the center part in the compact real form.
$\mathfrak{k}_1 \coloneqq \mathfrak{k} \cap \mathfrak{g}_1$. Hermitian metric on $\mathfrak{g}_1$ restricts to usual metric on $\mathfrak{k}_1$.
$(\mathfrak{k}_1)_{\mathbb{C}} = \mathfrak{g}_1$: let $Z = X + iY \in \mathfrak{g}_1 \subset \mathfrak{g}$ for $X, Y \in \mathfrak{k}$. Since $\mathfrak{z}$ is invariant under conjugation, both $X, Y \in \mathfrak{g}_1$. Hence $X, Y \in \mathfrak{k}_1$.
Take K_1 to be the image of K of $\mathrm{Ad}: K \to GL(\mathfrak{k})$. It is compact.
$\mathrm{Lie}(K_1) \cong \mathfrak{k}_1$: let $[Z] \in \mathrm{Lie}(K_1)$ where $Z \in \mathrm{Lie}(\mathrm{K}) = \mathfrak{k}$. $Z = Z^{\mathfrak{g}_1} + Z^{\mathfrak{z}}$.
$Z^{\mathfrak{g}_1} \in \mathfrak{k}_1$: $\overline{Z^{\mathfrak{g}_1}} = \bar{Z}^{\mathfrak{g}_1} = Z^{\mathfrak{g}_1}$ and hence $Z^{\mathfrak{g}_1} \in \mathfrak{k} \cap \mathfrak{g}_1 = \mathfrak{k}_1$.
$[Z] \mapsto Z^{\mathfrak{g}_1}$ is well-defined: $Z \in \mathrm{Ker}(\mathrm{ad})$ if and only if $Z \in \mathfrak{z}$, and so $Z^{\mathfrak{g}_1} = 0$.
It is injective for the same reason.
Surjectivity: for $Z \in \mathfrak{k}_1 \subset \mathfrak{k}$, $[Z] \mapsto Z$.
QED

Prop. If K is simply-connected compact, then $\mathfrak{k}_{\mathbb{C}}$ is semi-simple.
Proof: Need to see $\mathfrak{k}$ has trivial center. Let $\mathfrak{z}$ be the center and decompose by metric: $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{z}$. Since K is simply-connected, accordingly $K = K_1 \times Z$ where Z is commutative, which must then by $\mathbb{R}^n$. But K is compact and so $n = 0$, forcing $\mathfrak{z} = 0$.
QED

Decomposition in Lie algebra leads to decomposition of a simply-connected Lie group:

Consider the projection homomorphisms which correspond to Lie group homomorphisms. The identity components of kernels give the factors which are closed connected.
The two factors commute. Then have homomorphism from their product to G. It has inverse since the corresponding Lie algebra map has inverse.

Prop. Semi-simple $\mathfrak{g} = \bigoplus_{j=1}^{m} \mathfrak{g}_j$ where each $\mathfrak{g}_j$ is simple (no non-trivial ideal and $\dim \mathfrak{g}_j \geq 2$). This decomposition is unique (up to reordering).
Proof:
Suppose $\mathfrak{g}$ is not simple, and so it has a non-trivial ideal $\mathfrak{h}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. ($[\mathfrak{h}, \mathfrak{h}^{\perp}] \subset \mathfrak{h} \cap \mathfrak{h}^{\perp} = \{0\}$.) An ideal of $\mathfrak{h}$ is also an ideal of $\mathfrak{g}$. Repeating and we get a decomposition of $\mathfrak{g}$ into into simple ideals. ($\dim \mathfrak{g}_j \geq 2$ since otherwise it is center.)

Unique: each $\mathfrak{g}_j$ is an irreducible representation of $\mathfrak{g}$ (by adjoint action). Any morphism $\mathfrak{g}_j \to \mathfrak{g}_k$ cannot be an isomorphism and hence zero for $j \neq k$: there is $X, Y \in \mathfrak{g}_j$ with $[X, Y] \neq 0$, but $\left[\mathfrak{g}_j, \mathfrak{g}_k\right] = 0$.
QED

From now on always assume semi-simple.

Cartan subalgebra $\mathfrak{h}$:
1. $[\mathfrak{h}, \mathfrak{h}] = 0$. (**commutative**)
2. If $[H, X] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$. (**maximal commutative**)
3. ad_H is diagonalizable for all $H \in \mathfrak{h}$.

Construction: take a maximal commutative subalgebra $\mathfrak{t}$ in $\mathfrak{k}$. Take $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$.
$\mathfrak{h}$ is Cartan:
1, 2 are direct from definition.
3: for any $X \in \mathfrak{k}$, ad_X is skew-adjoint (with respect to the invariant metric) and hence diagonalizable. For $H = H_1 + iH_2 \in \mathfrak{h}$, since $[H_1, H_2] = 0$, they are simultaneously diagonalizable and so ad_H is diagonalizable.

Rank: dimension of a Cartan subalgebra.
Cartan subalgebra is unique up to automorphism of $\mathfrak{g}$ (proof skipped). Hence rank is well-defined.

Root $\boldsymbol{\alpha} \in \boldsymbol{i}\,\mathbf{t}^* =: \mathfrak{h}_{\mathbb{R}}^* \subset \mathfrak{h}^*$:
The adjoint K-action on $\mathfrak{k}$ is unitary. Hence $\mathfrak{t} \subset \mathfrak{k}$ acts on $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ as skew self-adjoint operators. Thus eigenvalues of $H \in \mathfrak{t}$ are purely imaginary.

(Note that the Hermitian metric restrict to be usual metric on $\mathfrak{h}_{\mathbb{R}} := it$.)

Simultaneous eigenspace decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha .$$

$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ (where $\mathfrak{g}_0 = \mathfrak{h}$):
By Jacobi identity. For $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$,
$[H,[X,Y]] = [X,[H,Y]] + [[H,X],Y] = (\beta, H)[X,Y] + (\alpha, H)[X,Y] = (\alpha + \beta, H)[X,Y]$.

If $\alpha \in R$, then $-\alpha \in R$:
$\bar{X} \in \mathfrak{g}_{-\alpha}$ if $X \in \mathfrak{g}_\alpha$ since α is purely imaginary valued.

R spans $\mathfrak{h}^*$:
$\mathfrak{g}$ has trivial center. If $H \in (\text{Span } R)^\perp \subset \mathfrak{h}$, then $[H,X] = \alpha(H)X = 0$ for $X \in \mathfrak{g}_\alpha$ for all $\alpha \in R$. Thus H is in the center and must be zero.

Theorem:
For each root α, there exists $H_\alpha \in \mathfrak{h}_{\mathbb{R}}$ and $H_\alpha \in \mathbb{R} \cdot \alpha$ (by identifying $\mathfrak{h}_{\mathbb{R}} \cong \mathfrak{h}_{\mathbb{R}}^*$ via the invariant metric), $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $H_\alpha, X_\alpha, Y_\alpha$ satisfy the $\mathfrak{sl}(2,\mathbb{C})$ relations.
Y_α can be taken to be $-\overline{X_\alpha}$.

Indeed $(\alpha, H_\alpha) = 2$ since $[H_\alpha, X_\alpha] = 2X_\alpha$.
Hence $\boldsymbol{H_\alpha} = \frac{2\boldsymbol{\alpha}}{\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle}$ **(called the coroot)** ($\alpha \in \mathfrak{h}_{\mathbb{R}}^*$ identified as $\mathfrak{h}_{\mathbb{R}}$) is the unique choice.

Proof:
Take H_α as above, $X \in \mathfrak{g}_\alpha - \{0\}$ and $Y = -\bar{X} \in \mathfrak{g}_{-\alpha}$ (since α is purely imaginary).
Then $[H_\alpha, X] = 2X$ and $[H_\alpha, Y] = -2Y$.
We know that $[X,Y] \in \mathfrak{h}$.

$\langle [\boldsymbol{X}, \boldsymbol{Y}], \boldsymbol{H} \rangle = (\boldsymbol{\alpha}, \boldsymbol{H}) \langle \boldsymbol{Y}, -\bar{\boldsymbol{X}} \rangle$ for any $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_{-\alpha}, H \in \mathfrak{h}$:
$\langle [X,Y], H \rangle = \langle Y, \text{ad}_{-\bar{X}} H \rangle = \langle Y, (\alpha, H)(-\bar{X}) \rangle = (\alpha, H) \langle Y, -\bar{X} \rangle$.

Hence in above $\langle [X,Y], H \rangle = (\alpha, H)|Y|^2 = \langle \alpha, H \rangle |Y|^2$ (identifying $\alpha \in \mathfrak{h}_{\mathbb{R}}$).
Thus $[X,Y]$ is perpendicular to any $H \in \alpha^\perp$, and so $[X,Y] \in \mathbb{C} \cdot H_\alpha$.
$\langle [X,Y], H_\alpha \rangle = 2|Y|^2$. $[X,Y] = \frac{2|Y|^2}{|H_\alpha|^2} H_\alpha$.
Thus if we take X with $|X| = |H_\alpha|/\sqrt{2}$ in the beginning, then $[X,Y] = H_\alpha$.

Prop: The only roots which are multiples of α is $\pm\alpha$.
Also $\mathfrak{g}_\alpha$ is one dimensional.
Proof:
Suppose $\beta = c\alpha$ is also a root.
$ad_{H_\alpha} X_\beta = (\beta, H_\alpha) X_\beta = 2c X_\beta$.
Since $\mathrm{Span}\{H_\alpha, X_\alpha, Y_\alpha\} \cong \mathfrak{sl}(2,\mathbb{C})$, 2c is an integer.
Reversing α and β, 2/c is also an integer.
Then c can only be $\pm\frac{1}{2}, \pm 1, \pm 2$.

Take α to be the shortest one among all roots in its direction.
Take $V^\alpha \subset \mathfrak{g}$ spanned by H_α and all $\mathfrak{g}_\beta$ where $\beta = c\alpha$ are roots where $c = \pm 1, \pm 2$.
V^α is invariant under $\mathfrak{s}_\alpha := \mathrm{Span}\{H_\alpha, X_\alpha, Y_\alpha\}$:
$[X_\alpha, X_\beta] \in \mathfrak{g}_{\alpha+\beta}$, $[X_\alpha, Y_\beta] \in \mathfrak{g}_{\alpha-\beta}$ where $\alpha \pm \beta$ are multiples of α.
Then $\mathfrak{s}_\alpha^\perp \subset V^\alpha$ is a representation of $\mathfrak{s}_\alpha$.
$\mathfrak{s}_\alpha^\perp$ is spanned by weight vectors $X \in \mathfrak{g}_\beta$ with even weights:
$ad_{H_\alpha} X = (\beta, H_\alpha) X = 2c\, X$. So the weight is either $\pm 2, \pm 4$.
For $\mathfrak{sl}(2,\mathbb{C})$ representation, this implies it also has the weight zero, a contradiction unless $\mathfrak{s}_\alpha^\perp = 0$.
Hence $V^\alpha = \mathfrak{s}_\alpha$, meaning $X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha}$ are the only root vectors (up to multiple) in the direction of α.
QED

For any roots α, β,
$$(\boldsymbol{\beta}, \boldsymbol{H_\alpha}) = \frac{\mathbf{2}\langle\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle}{\langle\boldsymbol{\alpha}, \boldsymbol{\alpha}\rangle} \in \mathbb{Z}:$$
$ad_{H_\alpha} X = (\beta, H_\alpha)\, X$ for $X \in \mathfrak{g}_\beta$.
$\mathfrak{g}$ is a representation of $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2,\mathbb{C})$. Hence $(\beta, H_\alpha) \in \mathbb{Z}$.

Hence $\boldsymbol{s_\alpha} \cdot \boldsymbol{\beta} - \boldsymbol{\beta} = \boldsymbol{k\alpha}$ **for** $\boldsymbol{k} \in \mathbb{Z}$.
$\boldsymbol{s_\alpha}(\boldsymbol{v}) := \boldsymbol{v} - \frac{\mathbf{2}\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle}{\langle\boldsymbol{\alpha},\boldsymbol{\alpha}\rangle}\boldsymbol{\alpha}$ is the reflection about $\alpha^\perp \subset \mathfrak{h}_\mathbb{R}^*$.

Weyl group:
$\langle s_\alpha : \alpha \in R\rangle \subset O(\mathfrak{h}_\mathbb{R}^*)$.

Prop: Weyl group preserves the set of roots.
For $Z \in \mathfrak{g}_\beta$, need to make a root vector in $\mathfrak{g}_{s_\alpha \cdot \beta}$.

For $\mathfrak{sl}(2,\mathbb{C}) = \langle H, X, Y\rangle$ representation π,
$\boldsymbol{U\pi(H)U^{-1} = -\pi(H)}$ **where** $\boldsymbol{U = e^{\pi(X)}e^{-\pi(Y)}e^{\pi(X)}}$:
$e^{\pi(X)}\pi(H)e^{-\pi(X)} = \mathrm{Ad}_{e^{\pi(X)}} \cdot \pi(H) = \exp \mathrm{ad}_{\pi(X)} \cdot \pi(H) = \pi(H) - 2\pi(X)$.
$e^{-\pi(Y)}\pi(H)e^{\pi(Y)} = \pi(H) + 2\pi(Y)$.
$e^{-\pi(Y)}\pi(X)e^{\pi(Y)} = \pi(X) + \pi(H) + \pi(Y)$.
$e^{-\pi(Y)}\big(\pi(H) - 2\pi(X)\big)e^{\pi(Y)} = -\pi(H) - 2\pi(X)$.
$e^{\pi(X)}\big(-\pi(H) - 2\pi(X)\big)e^{-\pi(X)} = -\pi(H) + 2\pi(X) - 2\pi(X) = -\pi(H)$.

Restricting $\mathfrak{g}$ as a representation of $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2,\mathbb{C})$, have
$U_\alpha \mathrm{ad}_{H_\alpha} U_\alpha^{-1} = -\mathrm{ad}_{H_\alpha}$.
Also for $H \in \alpha^\perp$, $[H, X_\alpha] = [H, Y_\alpha] = 0$. Hence $e^{\mathrm{ad}_{X_\alpha}} \cdot \mathrm{ad}_H = \mathrm{ad}_H = \mathrm{ad}_H \cdot e^{-\mathrm{ad}_{X_\alpha}}$.
Thus $U_\alpha \mathrm{ad}_H U_\alpha^{-1} = \mathrm{ad}_H$.
Combining, for all $H \in \mathfrak{h}_\mathbb{R}$,
$U_\alpha \mathrm{ad}_H U_\alpha^{-1} = \mathrm{ad}_{s_\alpha \cdot H}$.

Then $U_\alpha \cdot Z \in \mathfrak{g}_{s_\alpha \cdot \beta}$:
$ad_H \cdot (U_\alpha \cdot Z) = U_\alpha \cdot (U_\alpha^{-1} \cdot ad_H \cdot U_\alpha) \cdot Z = U_\alpha \cdot \mathrm{ad}_{s_\alpha \cdot H} \cdot Z = (\beta, s_\alpha \cdot H)\, U_\alpha \cdot Z$
$= (s_\alpha \cdot \beta, H)\, U_\alpha \cdot Z$.
QED

Weyl group is finite:
Its action on the set of roots gives an injection to the permutation group.
(It is an injection since Span $R = \mathfrak{h}_\mathbb{R}^*$.)

Summary for the root system $R \subset \mathfrak{h}_\mathbb{R}^* = i\,\mathfrak{t}^*$:
1. R spans $\mathfrak{h}_\mathbb{R}^*$.
2. For $\alpha \in R$, $\pm\alpha$ are the only multiples of α which belong to R.
3. $s_\alpha \cdot \beta \in R$ for $\alpha, \beta \in R$.
4. $\dfrac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$.

Theorem: $\mathfrak{g}$ is simple <=> R is irreducible.
(R irreducible means $R \neq R_1 \cup R_2$ where $\mathrm{Span}_\mathbb{R}(R_1)$ and $\mathrm{Span}_\mathbb{R}(R_2)$ are orthogonal.
Obviously in such a case R_1 and R_2 are root systems.)
Proof:
Semi-simple $\mathfrak{g}$ is direct sum of simple $\mathfrak{g}_k$.
Each $\boldsymbol{\mathfrak{g}_k}$ **is invariant under conjugation**:

$$\mathfrak{g} = \bigoplus_k \mathfrak{g}_k = \bigoplus_k \overline{\mathfrak{g}_k}.$$

By uniqueness $\overline{\mathfrak{g}_k} = \mathfrak{g}_l$. Suppose $k \neq l$. Then $\mathfrak{g}_k \cap \overline{\mathfrak{g}_k} = \{0\}$ and $[\mathfrak{g}_k, \overline{\mathfrak{g}_k}] = 0$.
$\mathfrak{g}_k \oplus \overline{\mathfrak{g}_k}$ is invariant under conjugation.
Then $(\mathfrak{g}_k \oplus \overline{\mathfrak{g}_k}) \cap \mathfrak{k}$ is an ideal of $\mathfrak{k}$. It leads to a decomposition of $\mathfrak{k}$ and hence a decomposition of the simply connected compact K.
Thus $(\mathfrak{g}_k \oplus \overline{\mathfrak{g}_k}) \cap \mathfrak{k} = \mathrm{Re}\, \mathfrak{g}_k$ corresponds to a compact subgroup $K_1 \subset K$.
$\mathrm{Re}\, \mathfrak{g}_k \cong \mathfrak{g}_k$ which is (real) Lie algebra isomorphism.
$(X + \bar{X} \leftrightarrow X.\ [X + \bar{X}, Y + \bar{Y}] = [X, Y] + \overline{[X, Y]} \leftrightarrow [X, Y].)$
Then $\mathrm{Lie}(K_1) \cong \mathfrak{g}_k$ has a complex structure!

Lie algebra $\mathfrak{k}$ of a compact noncommutative Lie group K can never has a complex structure:
Suppose it has a complex structure J (under which the Lie bracket is complex linear). Let H not in the center. ad_H is non-zero and skew-symmetric (with respect to a K-invariant BILINEAR metric such that J is isometry) and hence has an eigenvalue $i\lambda \neq 0$ and eigenvector X.
$[H, X] = \lambda JX$ and $[H, JX] = -\lambda X$.
So $[-\lambda JH, X] = \lambda^2 X$.
$\langle \lambda^2 X, X \rangle = \langle X, \mathrm{ad}_{\lambda JH} X \rangle = \langle X, -\lambda^2 X \rangle$!

Thus we have $\mathfrak{k}_k = \mathfrak{g}_k \cap \overline{\mathfrak{g}_k}$ and $\mathfrak{k} = \bigoplus \mathfrak{k}_k$. It corresponds to Lie group decomposition of K and they give compact real forms of $\mathfrak{g}_k$.
And correspondingly $\mathfrak{t} = \bigoplus \mathfrak{t}_k$ where $\mathfrak{t}_k = \mathfrak{k}_k \cap \mathfrak{t}$ which are maximal commutative subalgebra of $\mathfrak{k}_k$. $\mathfrak{h} = \bigoplus \mathfrak{h}_k$ where $\mathfrak{h}_k = (\mathfrak{t}_k)_{\mathbb{C}}$ which are Cartan subalegbras of $\mathfrak{g}_k$.

Then we have roots R_k of $\mathfrak{g}_k$ in $\mathfrak{h}_k^*$ (and Span $R_k = \mathfrak{h}_k^*$).
They are regarded as roots of $\mathfrak{g}$.

These are all the roots of $\mathfrak{g}$ in $\mathfrak{h}^* = \bigoplus \mathfrak{h}_k^*$ and hence $R = \cup_k R_k$:
We have a root space decomposition of $\mathfrak{g}$ by a direct sum of the root space decompositions of $\mathfrak{g}_k$.
QED

Exercises. (Section 7.8)

1. Let $\mathfrak{h}$ be the Lie algebra of complex 3×3 upper triangular matrices with zeros on the diagonal. Show that it does not have any Cartan subalgebra.
2. Give an example of a maximal commutative subalgebra of $\mathfrak{sl}(2, \mathbb{C})$ which is not a Cartan subalgebra.