Abstract root system

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Root system $R \subset E$:

1. R spans E .
2. For $\alpha \in R$, $\pm\alpha$ are the only multiples of α which belong to R.
3. $s_\alpha \cdot \beta \in R$ for $\alpha, \beta \in R$.
4. $\dfrac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$.

Weyl group: the group generated by s_α.
Can be identified as a subgroup of the permutation group of R (and hence is finite if R is finite).

If $R \subset E$ and $S \subset F$ are root systems, then so is $R \cup S \subset E \bigoplus F$.

Morphism of root system:
linear map A with $A(R) \subset S$ and commute with Weyl action:
$A(s_\alpha \cdot \beta) = s_{A\alpha} \cdot (A\beta)$.
Note that it may not preserve metric. (Allow scaling.)

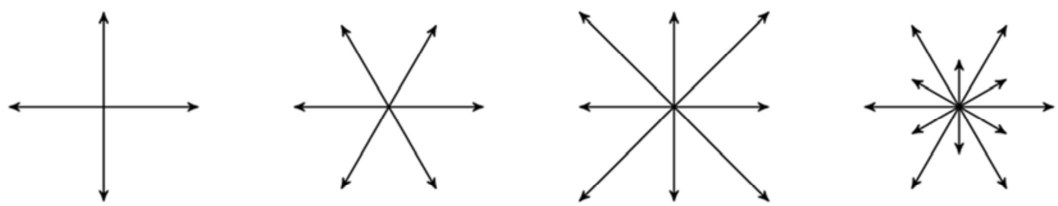
Prop. let α, β be linearly independent roots.
WLOG let $|\alpha| \geq |\beta|$. Then either

1. $\langle\alpha,\beta\rangle = 0$.
2. $\langle\alpha,\alpha\rangle = \langle\beta,\beta\rangle$ and the angle between the two lines is $\frac{\pi}{3}$.
3. $\langle\alpha,\alpha\rangle = 2\langle\beta,\beta\rangle$ and the angle between the two lines is $\frac{\pi}{4}$.
4. $\langle\alpha,\alpha\rangle = 3\langle\beta,\beta\rangle$ and the angle between the two lines is $\frac{\pi}{6}$.

Proof:
$m_1 = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ and $m_2 = \frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$.
$m_1 m_2 = 4\cos^2\theta$.
Hence $0 \leq m_1 m_2 \leq 4$.
Five cases: $m_1 m_2 = 0,1,2,3,4$. Remaining is plane geometry.

QED

Cor:
Angle between roots α and β is
strictly obtuse => $\alpha + \beta$ is root.
strictly acute => $\alpha - \beta$ and $\beta - \alpha$ are roots.
Proof:
Consider $s_\alpha \cdot \beta$ which is a root. QED



$A_1 \times A_1$ A_2 $B_2 \cong C_2$ G_2

R$^\vee$: set of all coroots $H_\alpha = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$.
Prop. $R^\vee$ is also a root system and it has the same Weyl group.
$(R^\vee)^\vee = R$.
Proof:
Condition 1 and 2 for root system are obvious.
Direct check that

$$\frac{2H_\alpha}{\langle H_\alpha, H_\alpha\rangle} = \alpha.$$

(Hence $(R^\vee)^\vee = R$.)

$$\frac{2\langle H_\alpha, H_\beta\rangle}{\langle H_\alpha, H_\alpha\rangle} = \langle \alpha, H_\beta\rangle = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}.$$

$$s_{H_\alpha} \cdot H_\beta = s_\alpha \cdot H_\beta = \frac{2}{\langle\beta,\beta\rangle} s_\alpha \cdot \beta = \frac{2}{\langle s_\alpha\cdot\beta, s_\alpha\cdot\beta\rangle} s_\alpha \cdot \beta = H_{s_\alpha\cdot\beta}.$$

QED

Base Δ (set of all positive simple roots):
a subset of R which is a basis of E, and every root is an integer

combination in Δ with coefficients of the same sign.

Any two vectors in Δ have right or obtuse angle:
Otherwise $\alpha - \beta$ would be a root, contradicting that all coefficients are the same sign.

Construction of a base:
Take a hyperplane not containing any root of R (take H not in any $\alpha^\perp$, and take $H^\perp$).
Take R^+ to be the roots in one side.

Prop. The set **Δ of indecomposible elements of R^+ is a base**.
(Indecomposible means $\alpha \neq \beta + \gamma$ for any $\beta, \gamma \in R^+$.)
Proof:
Any roots in R^+ is an integer combination of Δ with positive coefficients:
Suppose not. Take such a root α with minimal $\langle \alpha, H \rangle$.
$\alpha \notin \Delta$, so $\alpha = \beta + \gamma$ for $\beta, \gamma \in R^+$. But $\langle \alpha, H \rangle = \langle \beta, H \rangle + \langle \gamma, H \rangle$, contradicting the minimality.

Any other roots are $-\alpha$ for $\alpha \in R^+$. Hence **any root is an integer combination of Δ with coefficients of the same sign**.

Δ is linearly independent:
Suppose $\sum_\alpha c_\alpha \alpha = \sum_\beta d_\beta \beta$ where the coefficients are all positive (and the sums are over disjoint subsets of Δ).
Consider its norm squared: $\sum c_\alpha d_\beta \langle \alpha, \beta \rangle$.

$\langle \alpha, \beta \rangle \leq 0$ for any distict $\alpha, \beta \in \Delta$, and so the above has to be zero:
Otherwise $\alpha - \beta$ and $\beta - \alpha$ would be roots, and one of them belongs to R^+, contradicting that coefficients have the same sign.

Thus $\sum_\alpha c_\alpha \alpha = \sum_\beta d_\beta \beta = 0$ and all coefficients are positive. But all α

and β are in one side of H, and so this is impossible.
QED

Any base must arise in this way, namely, there is a hyperplane not containing any roots such that the base is the set of indecomposible elements in one side of the hyperplane:
Take an element h in the dual cone $\{h \in E^*: (h, \alpha) > 0 \; \forall \alpha \in \Delta\}$. Then Δ and R^+ is contained in one side of $h^\perp$. R^- is contained in the other side.
Taking the indecomposable roots in the positive side of $h^\perp$ gives a base. This is Δ: both are base and hence have the same number of elements. $\alpha \in \Delta$ is indecomposable: suppose $\alpha = \beta + \gamma$ for $\beta, \gamma \in R^+$. Expressing in terms of the basis Δ, we see that it is not possible.

Prop. If Δ is a base for R, then $\Delta^\vee$ is a base for $R^\vee$.
Proof: From above Δ arises as indecomposible roots on one side of a hyperplane. $\alpha^\vee$ for $\alpha \in R^+$ lie on the same side, and that for $\alpha \in R^-$ lie on the other side. Thus indecomposible coroots on the positive side gives rise to a base for $R^\vee$. This is $\Delta^\vee$: they have the same number of elements. $H_\alpha \in \Delta^\vee$ is indecomposible: the coroots in $\Delta^\vee$ also form a basis (since they just differ from roots by scaling). Suppose $H_\alpha = H_\beta + H_\gamma$ for $\beta, \gamma \in R^+$. Expressing in terms of the basis $\Delta^\vee$, we see that it is not possible.

Weyl chambers:
Connected components of $E - \bigcup_\alpha \alpha^\perp$.

Dominant (or fundamental) chamber C (relative to Δ):
$\langle \alpha, H \rangle > 0$ any $H \in C$ and $\alpha \in \Delta$.

{Base Δ} <-> {Weyl chamber C}:
-> take the dominant chamber relative to Δ.
It is the **dual cone** $\{H \in E^*: (\alpha, H) > 0 \; \forall \alpha \in \Delta\}$ which is a Weyl chamber since $(\alpha, H) \neq 0$ for any $\alpha \in R$.

<- Take any $H \in C$. $H^{\perp}$ does not contain any root and hence the indecomposible roots on the side $(\alpha, H) > 0$ define a base.
For all other elements $H' \in C$, since H' and H are in the same connected component of $E - \bigcup_{\alpha} \alpha^{\perp}$, (α, H_t) can never be zero and hence cannot change sign for a path H_t connecting them.

Exercises. (Section 7.8)

7. Suppose A is an isomorphism between two irreducible root systems. Show that it is a constant multiple of an isometry.
9. $$P(H) \coloneqq \prod_{\alpha \in R^+} \langle \alpha, H \rangle \, .$$
   Show that $P(w \cdot H) = \det(w) \, P(H)$ for all $w \in W$ and $H \in E$.