Root system $R \subset E$:
1. $R$ spans $E$.
2. For $\alpha \in R$, $\pm \alpha$ are the only multiples of $\alpha$ which belong to $R$.
3. $s_\alpha \cdot \beta \in R$ for $\alpha, \beta \in R$.
4. $\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

Weyl group: the group generated by $s_\alpha$.
Can be identified as a subgroup of the permutation group of $R$ (and hence is finite if $R$ is finite).

If $R \subset E$ and $S \subset F$ are root systems, then so is $R \cup S \subset E \oplus F$.

Morphism of root system:
linear map $A$ with $A(R) \subset S$ and commute with Weyl action:
$A(s_\alpha \cdot \beta) = s_{A\alpha} \cdot (A\beta)$.
Note that it may not preserve metric. (Allow scaling.)

Prop. let $\alpha, \beta$ be linearly independent roots.
WLOG let $|\alpha| \geq |\beta|$. Then either
1. $\langle \alpha, \beta \rangle = 0$.
2. $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$ and the angle between the two lines is $\frac{\pi}{3}$.
3. $\langle \alpha, \alpha \rangle = 2\langle \beta, \beta \rangle$ and the angle between the two lines is $\frac{\pi}{4}$.
4. $\langle \alpha, \alpha \rangle = 3\langle \beta, \beta \rangle$ and the angle between the two lines is $\frac{\pi}{6}$.

Proof:
$m_1 = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ and $m_2 = \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$.
$m_1 m_2 = 4 \cos^2 \theta$.
Hence $0 \leq m_1 m_2 \leq 4$.
Five cases: $m_1 m_2 = 0, 1, 2, 3, 4$. Remaining is plane geometry.
QED

**Cor:**
Angle between roots $\alpha$ and $\beta$ is
strictly obtuse $\Rightarrow$ $\alpha + \beta$ is root.
strictly acute $\Rightarrow$ $\alpha - \beta$ and $\beta - \alpha$ are roots.

**Proof:**
Consider $s_\alpha \cdot \beta$ which is a root. QED

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\[ R^\vee: \text{set of all coroots } H_\alpha = \frac{2 \alpha}{\langle \alpha, \alpha \rangle}. \]

**Prop.** $R^\vee$ is also a root system and it has the same Weyl group.
$(R^\vee)^\vee = R$.

**Proof:**
Condition 1 and 2 for root system are obvious.
Direct check that
\[ 2H_\alpha \over \langle H_\alpha, H_\alpha \rangle = \alpha. \]
(Hence $(R^\vee)^\vee = R$.)
\[ 2\langle H_\alpha, H_\beta \rangle \over \langle H_\alpha, H_\alpha \rangle = \langle \alpha, H_\beta \rangle = \frac{2 \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}. \]
\[ s_{H_\alpha} \cdot H_\beta = s_\alpha \cdot H_\beta = \frac{2}{\langle \beta, \beta \rangle} s_\alpha \cdot \beta = \frac{2}{\langle s_\alpha \cdot \beta, s_\alpha \cdot \beta \rangle} s_\alpha \cdot \beta = H_{s_\alpha \cdot \beta}. \]
QED

**Base** $\Delta$ (set of all positive simple roots):
a subset of $R$ which is a basis of $E$, and every root is an integer
combination in $\Delta$ with coefficients of the same sign.

**Any two vectors in $\Delta$ have right or obtuse angle:**
Otherwise $\alpha - \beta$ would be a root, contradicting that all coefficients are the same sign.

**Construction of a base:**
Take a hyperplane not containing any root of $R$ (take $H$ not in any $\alpha^\perp$, and take $H^\perp$).
Take $R^+$ to be the roots in one side.

**Prop.** The set $\Delta$ of indecomposable elements of $R^+$ is a base.
(Indecomposable means $\alpha \neq \beta + \gamma$ for any $\beta, \gamma \in R^+$.)

**Proof:**
**Any roots in $R^+$ is an integer combination of $\Delta$ with positive coefficients:**
Suppose not. Take such a root $\alpha$ with minimal $\langle \alpha, H \rangle$.
$\alpha \notin \Delta$, so $\alpha = \beta + \gamma$ for $\beta, \gamma \in R^+$. But $\langle \alpha, H \rangle = \langle \beta, H \rangle + \langle \gamma, H \rangle$, contradicting the minimality.

Any other roots are $-\alpha$ for $\alpha \in R^+$. Hence **any root is an integer combination of $\Delta$ with coefficients of the same sign.**

**$\Delta$ is linearly independent:**
Suppose $\sum_\alpha c_\alpha \alpha = \sum_\beta d_\beta \beta$ where the coefficients are all positive (and the sums are over disjoint subsets of $\Delta$).
Consider its norm squared: $\sum c_\alpha d_\beta \langle \alpha, \beta \rangle$.

$\langle \alpha, \beta \rangle \leq 0$ for any distinct $\alpha, \beta \in \Delta$, and so the above has to be zero: Otherwise $\alpha - \beta$ and $\beta - \alpha$ would be roots, and one of them belongs to $R^+$, contradicting that coefficients have the same sign.

Thus $\sum c_\alpha \alpha = \sum d_\beta \beta = 0$ and all coefficients are positive. But all $\alpha$
and $\beta$ are in one side of $H$, and so this is impossible.

QED

**Any base must arise in this way**, namely, there is a hyperplane not containing any roots such that the base is the set of indecomposable elements in one side of the hyperplane:

Take an element $h$ in the dual cone $\{h \in E^* : (h, \alpha) > 0 \, \forall \alpha \in \Delta\}$. Then $\Delta$ and $R^+$ is contained in one side of $h^\perp$. $R^-$ is contained in the other side.

Taking the indecomposable roots in the positive side of $h^\perp$ gives a base. This is $\Delta$: both are base and hence have the same number of elements. $\alpha \in \Delta$ is indecomposable: suppose $\alpha = \beta + \gamma$ for $\beta, \gamma \in R^+$. Expressing in terms of the basis $\Delta$, we see that it is not possible.

**Prop.** If $\Delta$ is a base for $R$, then $\Delta^\vee$ is a base for $R^\vee$.

**Proof:** From above $\Delta$ arises as indecomposable roots on one side of a hyperplane. $\alpha^\vee$ for $\alpha \in R^+$ lie on the same side, and that for $\alpha \in R^-$ lie on the other side. Thus indecomposable coroots on the positive side gives rise to a base for $R^\vee$. This is $\Delta^\vee$: they have the same number of elements. $H_\alpha \in \Delta^\vee$ is indecomposable: the coroots in $\Delta^\vee$ also form a basis (since they just differ from roots by scaling). Suppose $H_\alpha = H_\beta + H_\gamma$ for $\beta, \gamma \in R^+$. Expressing in terms of the basis $\Delta^\vee$, we see that it is not possible.

**Weyl chambers:**
Connected components of $E - \bigcup_\alpha \alpha^\perp$.

**Dominant (or fundamental) chamber** $C$ (relative to $\Delta$): $\langle \alpha, H \rangle > 0$ any $H \in C$ and $\alpha \in \Delta$.

**{Base $\Delta$} <-> {Weyl chamber $C$}**:
- -> take the dominant chamber relative to $\Delta$.
It is the dual cone $\{H \in E^* : (\alpha, H) > 0 \, \forall \alpha \in \Delta\}$ which is a Weyl chamber since $(\alpha, H) \neq 0$ for any $\alpha \in R$. 
<- Take any $H \in C$. $H^\perp$ does not contain any root and hence the indecomposable roots on the side $(\alpha, H) > 0$ define a base. For all other elements $H' \in C$, since $H'$ and $H$ are in the same connected component of $E - \bigcup_\alpha \alpha^\perp$, $(\alpha, H_t)$ can never be zero and hence cannot change sign for a path $H_t$ connecting them.

**Exercises. (Section 7.8)**

7. Suppose $A$ is an isomorphism between two irreducible root systems. Show that it is a constant multiple of an isometry.

9. $P(H) := \prod_{\alpha \in R^+} \langle \alpha, H \rangle$.

Show that $P(w \cdot H) = \det(w) P(H)$ for all $w \in W$ and $H \in E$. 