Prop. Given a root, there exists a base containing it.
Proof: A base corresponds to a chamber. Given a root $\alpha$, there is a chamber which has a facet given by $\alpha^\perp$ (and $(H, \alpha) > 0$ for $H$ inside the chamber). $H^{>0}$ produces a base. $\alpha$ is indecomposable: We can take $H$ very close to the hyperplane $\alpha^\perp$ such that $(H, \alpha)$ is minimal among all positive roots. QED

Prop. The Weyl group $W$ is generated by $s_\alpha$ where $\alpha \in \Delta$. It acts faithfully and transitively on the set of Weyl chambers. (Hence as sets, $W \cong \{\text{Weyl chambers}\} \cong \{\text{Bases}\}$.)
Proof: Let $W' \subset W$ be generated by $s_\alpha$ for $\alpha \in \Delta$.
Let $C$ be the dominant chamber.
Want: for $H'$ in any chamber, there is $w \in W'$ such that $w \cdot H' \in C$.
Suppose $H'$ not in $C$. So there is a wall in between: there exists $\alpha \in \Delta$ such that $(\alpha, H') < 0$.
Reflection along this wall decreases the distance: Fix $H \in C$.

$$|H' - H|^2 - |s_\alpha \cdot H' - H|^2 = -\frac{4\langle \alpha, H' \rangle}{\langle \alpha, \alpha \rangle} \langle \alpha, H \rangle > 0.$$  

Keep on doing this, gradually $H'$ is reflected into $C$ since $W'$ is finite. Hence $W'$, and hence $W$, acts on Weyl chambers transitively. Faithfulness is obvious.

For any root $\beta \in R$, $\beta \in \Delta_{C'}$ for some chamber $C'$. By above there exists some $w \in W'$ such that $w \cdot C' = C$. Then $w \cdot \beta \in \Delta$.
$s_\beta = w^{-1} \cdot s_{w \cdot \beta} \cdot w \in W'$.
Hence $W = W'$. QED

Minimal expression: Write $w \in W$ in a minimal product of reflections associated to elements in $\Delta$.

Prop. Two distinct elements in $\overline{C}$ cannot lie in the same orbit of $W$.  

(Cont) Abstract root system
Proof: Want to say $H' \neq w \cdot H$ for any $w$. Induction on length of minimal expression.
Let $1 \neq w = s_{\alpha_1} \ldots s_{\alpha_k}$ be a minimal expression ($\alpha_i \in \Delta$).

Then $C$ and $w \cdot C$ are on different sides of $\alpha_1^\perp$:
Again use induction. Suppose $C$ and $s_{\alpha_1} \ldots s_{\alpha_k} \cdot C$ are on the same side. So $s_{\alpha_1} \ldots s_{\alpha_{k-1}} \cdot C$ is on another side by inductive assumption. Then $s_{\alpha_k} \cdot C$ and $C$ are on different sides of $(u^{-1} \cdot \alpha_1)^\perp$ where $u = s_{\alpha_1} \ldots s_{\alpha_{k-1}}$. But then $(u^{-1} \cdot \alpha_1)^\perp = \alpha_k^\perp$ and so $s_{\alpha_k} = s_{u^{-1} \cdot \alpha_1} = u^{-1} s_{\alpha_1} u$. Then $w = u \cdot s_{\alpha_k} = s_{\alpha_1} u = s_{\alpha_2} \ldots s_{\alpha_{k-1}}$, contradicting the minimality.

Suppose $H' = w \cdot H$. Then $H' \in \alpha_1^\perp$. Thus $H' = s_{\alpha_2} \ldots s_{\alpha_{k}} \cdot H$, contradicting the inductive assumption. QED

Prop: For $\alpha \in \Delta$, $s_{\alpha}$ preserves $R^+ - \{\alpha\}$.
Proof: Consider $\beta \in R^+ - \{\alpha\}$ and express it in terms of the base. It must involve an element $\gamma$ in the base which is not $\alpha$. $s_{\alpha} \cdot \beta = \beta - k\alpha$ and so it does not change the coefficient of $\gamma$, which is positive. Hence $s_{\alpha} \cdot \beta$ is still positive. QED

Dynkin diagram:
Vertices are base roots.
Number of edges between two vertices $\alpha, \beta$ is $\frac{\langle \alpha, \alpha \rangle}{\langle \beta, \beta \rangle}$ (WLOG $|\alpha| \geq |\beta|$) which is either $0,1,2,3$. (Recall that it determines the angle, which must be obtuse.)
Direction of edge is from longer to shorter.
(Choice of base does not matter: any two are related by reflection.)
**Morphism of Dynkin diagram:**
map between vertex sets preserving the numbers and directions of arrows
between any two vertices.

**R is irreducible <=> Dynkin diagram is connected:**
<= $R = R_1 \cup R_2$, then $\Delta = \Delta_1 \cup \Delta_2$ which are orthogonal to each other.
Then obviously the Dynkin diagram is disconnected.
=> If Dynkin disconnected, then $\Delta = \Delta_1 \cup \Delta_2$ which are orthogonal to each
other. All roots are obtained from base by Weyl action. Since orthogonal
the Weyl action preserves $E_i = \text{Span}(\Delta_i)$. Hence any root is either in $E_1$
or $E_2$.

**$R_1$ and $R_2$ are isomorphic <=> Dynkin diagrams are isomorphic:**
WLOG assume irreducible.
=> Take base of $R_1$, mapping to a base of $R_2$. Then the isomorphism is an
isometry up to scaling.
<= We have map between base roots, which is isometry up to scaling.
Then it certainly respects Weyl group actions.

**Classification:**

![Dynkin Diagrams](image)

**Integral structure:**
$E^*_\mathbb{Z} = \mathbb{Z} \cdot \{H_\alpha \in E^* \text{ for } \alpha \in \Delta\}$ gives the integral structure (which is a lattice
in $E^* = \mathfrak{h}_{\mathbb{R}}$).
The dual is $E^*_\mathbb{Z} = \{\mu \in E : (\mu, H_\alpha) \in \mathbb{Z}\}$. Recall
$$(\mu, H_\alpha) = \frac{2\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$ 
So $R \subset E^*_\mathbb{Z}$.
The dual basis $\{H^*_\alpha\} \subset E^*_\mathbb{Z}$ of $\{H_\alpha\} \subset E^*_\mathbb{Z}$ is called the fundamental weights.
It is characterized by
\[ H(\alpha, H_B) = \frac{2\langle H^*, \beta \rangle}{\langle \beta, \beta \rangle} = \delta_{\alpha \beta}. \]

**A special element:**
\[
\delta := \frac{1}{2} \sum_{\alpha \in R^+} \alpha.
\]
\((\delta, H_\alpha) = 1\) for all \(\alpha \in \Delta\) (and hence \(\delta \in E_{Z,+} = E_Z \cap C\)): 
\[
\frac{1}{2}(\alpha, H_\alpha) = 1.
\]
For other \(\beta \in R^+, s_\alpha \cdot \beta \in R^+\). If \(\beta \perp \alpha\), then \(\langle \beta, H_\alpha \rangle = 0\); if not, then \(\beta \neq s_\alpha \cdot \beta\) and \(\langle \beta + s_\alpha \cdot \beta, H_\alpha \rangle = 0\). Hence their contribution sum up to zero.

**Partial ordering:**
Like \(\mathfrak{sl}(3, \mathbb{C})\), have partial ordering on \(E\):
\[
\mu \geq \lambda \text{ if } \mu - \lambda \in \mathbb{R}_{\geq 0} \cdot \Delta.
\]

It has the following properties (proof skipped):

**If \(\mu \in \overline{C}\), then \(\mu \geq 0\).** \(w \cdot \mu \leq \mu \ \forall w \in W\).
**If \(\lambda \in \text{Conv}(W \cdot \mu)\) if and only if \(W \cdot \lambda \leq \mu\).**

**If \(\mu \in E_{Z,+} = E_Z \cap C\), then \(\mu \geq \delta\).**

**Exercises. (Section 8.12)**

1. Let \(\alpha, \beta \in R\) be linearly independent. If \(\alpha + k\beta \in R\) for \(k \in \mathbb{Z}_+\), then \(\alpha + l\beta \in R\) for \(l = 0, \ldots, k\).
10. Show that if \(-I \notin W\), then the Dynkin diagram must have a non-trivial automorphism.