(Cont) Abstract root system

Friday, April 6, 2018 8:40 PM

Prop. Given a root, there exists a base containing it.
Proof: A base corresponds to a chamber. Given a root α, there is a chamber which has a facet given by $\alpha^{\perp}$ (and $(H, \alpha) > 0$ for H inside the chamber). $H^{>0}$ produces a base. α is indecomposible: We can take H very close to the hyperplane $\alpha^{\perp}$ such that (H, α) is minimal among all positive roots. QED

Prop. The Weyl group W is generated by s_α where $\alpha \in \Delta$.
It acts faithfully and transitively on the set of Weyl chambers.
(Hence as sets, $W \cong$ {Weyl chambers} $\cong$ {Bases}.)
Proof:
Let $W' \subset W$ be generated by s_α for $\alpha \in \Delta$.
Let C be the dominant chamber.
Want: for H' in any chamber, there is $w \in W'$ such that $w \cdot H' \in C$.
Suppose H' not in C. So there is a wall in between: there exists $\alpha \in \Delta$ such that $(\alpha, H') < 0$.
Reflection along this wall decreases the distance: Fix $H \in C$.

$$|H' - H|^2 - |s_\alpha \cdot H' - H|^2 = -\frac{4\langle \alpha, H' \rangle}{\langle \alpha, \alpha \rangle} \langle \alpha, H \rangle > 0.$$

Keep on doing this, gradually H' is reflected into C since W' is finite.
Hence W', and hence W, acts on Weyl chambers transitively.
Faithfulness is obvious.

For any root $\beta \in R$, $\beta \in \Delta_{C'}$ for some chamber C'. By above there exists some $w \in W'$ such that $w \cdot C' = C$. Then $w \cdot \beta \in \Delta$.
$s_\beta = w^{-1} \cdot s_{w \cdot \beta} \cdot w \in W'$.
Hence $W = W'$. QED

Minimal expression: Write $w \in W$ in a minimal product of reflections associated to elements in Δ.

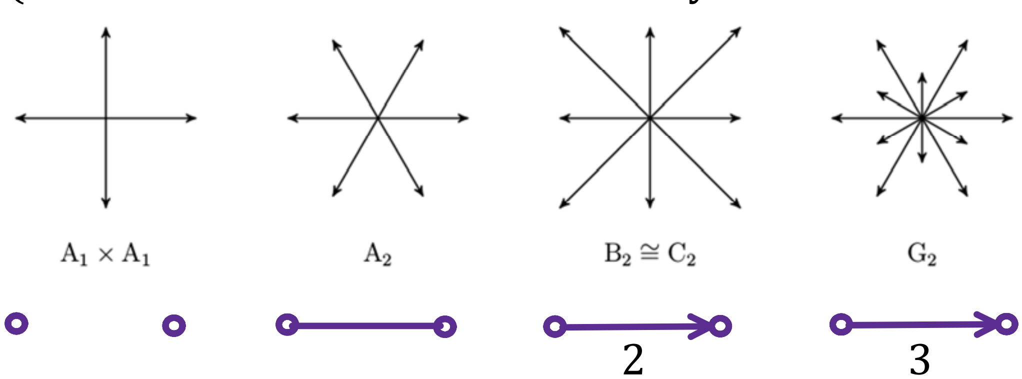
Prop. Two distinct elements in $\bar{C}$ cannot lie in the same orbit of W.

Proof: Want to say $H' \neq w \cdot H$ for any w. Induction on length of minimal expression.
Let $1 \neq w = s_{\alpha_1} \dots s_{\alpha_k}$ be a minimal expression ($\alpha_i \in \Delta$).
Then C and $w \cdot C$ are on different sides of $\alpha_1^\perp$:
Again use induction. Suppose C and $s_{\alpha_1} \dots s_{\alpha_k} \cdot C$ are on the same side. So $s_{\alpha_1} \dots s_{\alpha_{k-1}} \cdot C$ is on another side by inductive assumption. Then $s_{\alpha_k} \cdot C$ and C are on different sides of $(u^{-1} \cdot \alpha_1)^\perp$ where $u = s_{\alpha_1} \dots s_{\alpha_{k-1}}$. But then $(u^{-1} \cdot \alpha_1)^\perp = \alpha_k^\perp$ and so $s_{\alpha_k} = s_{u^{-1} \cdot \alpha_1} = u^{-1} s_{\alpha_1} u$. Then $w = u \cdot s_{\alpha_k} = s_{\alpha_1} u = s_{\alpha_2} \dots s_{\alpha_{k-1}}$, contradicting the minimality.

Suppose $H' = w \cdot H$. Then $H' \in \alpha_1^\perp$. Thus $H' = s_{\alpha_2} \dots s_{\alpha_k} \cdot H$, contradicting the inductive assumption. QED

Prop: For $\alpha \in \Delta$, s_α preserves $R^+ - \{\alpha\}$.
Proof: Consider $\beta \in R^+ - \{\alpha\}$ and express it in terms of the base. It must involve an element γ in the base which is not α. $s_\alpha \cdot \beta = \beta - k\alpha$ and so it does not change the coefficient of γ, which is positive. Hence $s_\alpha \cdot \beta$ is still positive. QED

Dynkin diagram:
Vertices are base roots.
Number of edges between two vertices α, β is $\frac{\langle \alpha,\alpha \rangle}{\langle \beta,\beta \rangle}$ (WLOG $|\alpha| \geq |\beta|$) which is either 0,1,2,3. (**Recall that it determines the angle, which must be obtuse.**)
Direction of edge is from longer to shorter.
(Choice of base does not matter: any two are related by reflection.)

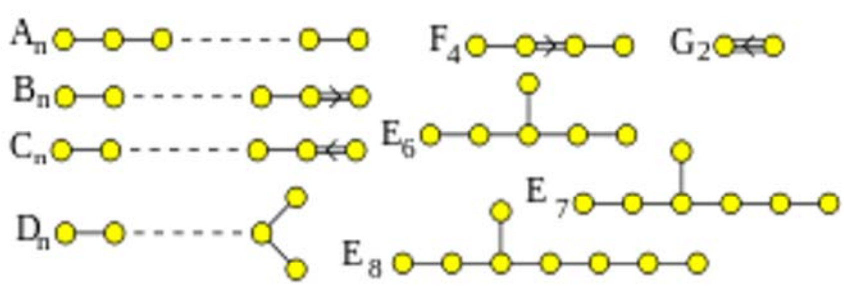


Morphism of Dynkin diagram:
map between vertex sets preserving the numbers and directions of arrows between any two vertices.

R is irreducible <=> Dynkin diagram is connected:
<= $R = R_1 \cup R_2$, then $\Delta = \Delta_1 \cup \Delta_2$ which are orthogonal to each other. Then obviously the Dynkin diagram is disconnected.
=> If Dynkin disconnected, then $\Delta = \Delta_1 \cup \Delta_2$ which are orthogonal to each other. All roots are obtained from base by Weyl action. Since orthogonal the Weyl action preserves $E_i = \mathrm{Span}(\Delta_i)$. Hence any root is either in E_1 or E_2.

R_1 and R_2 are isomorphic <=> Dynkin diagrams are isomorphic:
WLOG assume irreducible.
=> Take base of R_1, mapping to a base of R_2. Then the isomorphism is an isometry up to scaling.
<= We have map between base roots, which is isometry up to scaling. Then it certainly respects Weyl group actions.

Classification:



Integral structure:
$E^*_{\mathbb{Z}} = \mathbb{Z} \cdot \{H_\alpha \in E^* \text{ for } \alpha \in \Delta\}$ gives the integral structure (which is a lattice in $E^* = \mathfrak{h}_{\mathbb{R}}$).
The dual is $E_{\mathbb{Z}} = \{\mu \in E : (\mu, H_\alpha) \in \mathbb{Z}\}$. Recall

$$(\mu, H_\alpha) = \frac{2\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

So $R \subset E_{\mathbb{Z}}$.
The dual basis $\{H^*_\alpha\} \subset E_{\mathbb{Z}}$ of $\{H_\alpha\} \subset E^*_{\mathbb{Z}}$ is called the fundamental weights. It is characterized by

$$(H_\alpha^*, H_\beta) = \frac{2\langle H_\alpha^*, \beta\rangle}{\langle \beta, \beta\rangle} = \delta_{\alpha\beta}.$$

A special element:

$$\boldsymbol{\delta} := \frac{\mathbf{1}}{\mathbf{2}} \sum_{\boldsymbol{\alpha} \in \boldsymbol{R}^+} \boldsymbol{\alpha}\,.$$

$(\delta, H_\alpha) = 1$ for all $\alpha \in \Delta$ (and hence $\delta \in E_{\mathbb{Z},+} = E_{\mathbb{Z}} \cap C$):

$$\frac{1}{2}(\alpha, H_\alpha) = 1.$$

For other $\beta \in R^+$, $s_\alpha \cdot \beta \in R^+$. If $\beta \perp \alpha$, then $\langle \beta, H_\alpha \rangle = 0$; if not, then $\beta \neq s_\alpha \cdot \beta$ and $\langle \beta + s_\alpha \cdot \beta, H_\alpha \rangle = 0$. Hence their contribution sum up to zero.

Partial ordering:

Like $\mathfrak{sl}(3, \mathbb{C})$, have partial ordering on E:

$\mu \geq \lambda$ if $\mu - \lambda \in \mathbb{R}_{\geq 0} \cdot \Delta$.

It has the following properties (proof skipped):

If $\boldsymbol{\mu} \in \overline{\boldsymbol{C}}$, then $\boldsymbol{\mu} \geq \mathbf{0}$. $\boldsymbol{w} \cdot \boldsymbol{\mu} \leq \boldsymbol{\mu}\ \forall \boldsymbol{w} \in \boldsymbol{W}$.

$\boldsymbol{\lambda} \in \mathrm{Conv}(\boldsymbol{W} \cdot \boldsymbol{\mu})$ if and only if $\boldsymbol{W} \cdot \boldsymbol{\lambda} \leq \boldsymbol{\mu}$.

If $\boldsymbol{\mu} \in \boldsymbol{E}_{\mathbb{Z},+} = \boldsymbol{E}_{\mathbb{Z}} \cap \boldsymbol{C}$, then $\boldsymbol{\mu} \geq \boldsymbol{\delta}$.

Exercises. (Section 8.12)

1. Let $\alpha, \beta \in R$ be linearly independent. If $\alpha + k\beta \in R$ for $k \in \mathbb{Z}_+$, then $\alpha + l\beta \in R$ for $l = 0, \ldots, k$.
10. Show that if $-I \notin W$, then the Dynkin diagram must have a non-trivial automorphism.