Main theorem: \( \mathfrak{g} \) semi-simple.
{irreducible representations of \( \mathfrak{g} \) \( \leftrightarrow \) \( \mathbb{C} \cap \mathfrak{h}_{\mathbb{Z}}^{*} \)}
where \( \rightarrow \) is given by taking the highest weight.

Weights \( \mu \) are integral:
For any \( H_{\alpha} \) for \( \alpha \in \Delta \), restrict the representation to \( s_{\alpha} \cong \mathfrak{sl}(2, \mathbb{C}) \) and so \( (\mu, H_{\alpha}) \in \mathbb{Z} \).

The set of weights (and their multiplicities) is invariant under the Weyl group:
Recall for \( \mathfrak{sl}(2, \mathbb{C}) \) representation,
\( U\pi(H)U^{-1} = -\pi(H) \) where \( U = e^{\pi(X)}e^{-\pi(Y)}e^{\pi(X)} : V \to V \).
For \( s_{\alpha} \in W \), consider \( s_{\alpha} \cong \mathfrak{sl}(2, \mathbb{C}) \) and \( U_{\alpha} : V \to V \).

For \( v \in V_{\mu}, U_{\alpha} \cdot v \in V_{s_{\alpha}\mu} \):
\[
\pi(H_{\alpha}) \cdot (U_{\alpha} \cdot v) = U_{\alpha} \cdot (U_{\alpha}^{-1} \cdot \pi(H_{\alpha}) \cdot U_{\alpha}) \cdot v = U_{\alpha} -\pi(H_{\alpha})v
= (\mu, s_{\alpha} \cdot H_{\alpha}) U_{\alpha} \cdot v.
\]
If \( H \perp \alpha, H \) commutes with \( X_{\alpha} \) and \( Y_{\alpha} \) and hence \( U_{\alpha} \). Then
\[
\pi(H) \cdot (U_{\alpha} \cdot v) = (\mu, H) U_{\alpha} \cdot v = (\mu, s_{\alpha} \cdot H) U_{\alpha} \cdot v.
\]
Hence \( \pi(H) \cdot (U_{\alpha} \cdot v) = (\mu, s_{\alpha} \cdot H) U_{\alpha} \cdot v = (s_{\alpha} \cdot \mu, H) U_{\alpha} \cdot v \) for all \( H \).

Proof of Main Theorem:
Exactly like \( \mathfrak{sl}(3, \mathbb{C}) \), irreducible representations if and only if highest weight representations.
(\( \Rightarrow \) acting by positive root vectors until reaching the highest.
\( \Leftarrow \) semi-simple Lie algebra is completely reducible. Highest weight vector
(which has multiplicity one) must belong to one of the irreducible factor, but it is
 cyclic and hence there is only one factor.)

Then take the highest weight, which is dominant integral: for \( H_{\alpha} \) where \( \alpha \in R^{+} \),
consider \( s_{\alpha} \cong \mathfrak{sl}(2, \mathbb{C}) \) and restrict as its representation. A highest weight vector \( v \) of \( \pi \) is in particular a highest weight vector of \( \pi|s_{\alpha} \). Hence \( H_{\alpha} \cdot v = \lambda v \) where
\( \lambda \in \mathbb{Z}_{\geq 0} \).

Two highest weight representations with the same highest weight are
isomorphic by the same proof. (Consider the subspace of \( V \oplus W \) generated by
the highest weight vectors \( (v, w) \) which is again a highest weight representation.
The projection maps are isomorphisms.)

Now for \( \mu \in \mathfrak{h}_{\mathbb{Z}}^{*} \), need to cook up a \( \mu \)-highest weight representation.
Cook up a canonical representation (called Verma module) which is infinite
dimensional, and then take quotient! Do it below.

QED

Note: highest weight infinite-dimensional representation may NOT be
irreducible, and two with the same highest weight may not be isomorphic!
Also the highest weight can be any complex numbers!

\( \mathfrak{sl}(2, \mathbb{C}) \)-Verma module \( W_{\mu} \) with highest weight \( \mu \in \mathbb{C} \):
Take formal span of \( \{v_{0}, v_{1}, ... \} \). Define
\[
Y \cdot v_{j} = v_{j+1}, \quad H \cdot v_{j} = (\mu - 2j) v_{j}, \quad X \cdot v_{0} = 0, \quad X \cdot v_{j} = X \cdot Y_{j} \cdot v_{0} = Y \cdot X \cdot Y_{j-1} \cdot v_{0} + H \cdot v_{j-1} = Y \cdot X \cdot Y_{j-1} \cdot v_{0} + \mu(-2(j-1))v_{j-1} = \cdots = j(j - 1)(j - 2)\cdots(\mu + 1) \cdot v_{0}.
\]
If $\mu = m \in \mathbb{Z}_{\geq 0}$,
\[ X \cdot v_{m+1} = 0. \]
Thus $U_\mu = \text{Span}\{v_{m+1}, \ldots\}$ is invariant. The quotient $V_\mu$ is then a finite dimensional highest weight representation.

For general $\mathfrak{g}$, a basis of $W_\mu$ is given by $Y_1^{\mu_1} \cdots Y_N^{\mu_N} \cdot v_0$ where $v_0$ is a highest weight vector and $Y_i$ are all the negative roots in this order.

**Enveloping algebra $A$:**
An associative algebra with identity and $i: \mathfrak{g} \to A$ with $i([X,Y]) = i(X)i(Y) - i(Y)i(X)$, and $i(\mathfrak{g})$ generates $A$, that is, the smallest subalgebra with 1 containing $i(\mathfrak{g})$ is $A$.

**Universal enveloping algebra $U_\mathfrak{g}$:**
The enveloping algebra such that every other enveloping algebra is a quotient of $U_\mathfrak{g}$ (which is compatible with $i: \mathfrak{g} \to U_\mathfrak{g}$).

*ex. $\mathfrak{g} = \mathbb{C}$. Then $U_\mathfrak{g}$ is the free algebra $\langle H \rangle$.*

*ex. $\mathfrak{sl}(2, \mathbb{C})$. $U_\mathfrak{g} = \langle X, Y, H \rangle / \langle XY - YX - H, HX - XH - 2X, HY - YH + 2Y \rangle$.

The above $\langle X, Y, H \rangle$ is the tensor algebra (which is free algebra quotient by bilinear relations).

**Construction:**
Take tensor algebra (treating $\mathfrak{g}$ as a vector space)
\[ T(\mathfrak{g}) := \bigoplus_{k=0}^{\infty} \mathfrak{g}^\otimes k. \]
Then define $U_\mathfrak{g}$ as the quotient by $\langle XY - YX - [X,Y]: X, Y \in \mathfrak{g} \rangle$.

**It is universal:**
If $i: \mathfrak{g} \to A$, then have $T(\mathfrak{g}) \to A$ (mapping 1 to 1) which is surjective. If $i([X,Y]) = i(X)i(Y) - i(Y)i(X)$, then the map descends to $U_\mathfrak{g} \to A$.

In particular, representation $\mathfrak{g} \to A = \text{End}(V)$ (where $V$ can be infinite-dimensional) is one-to-one corresponding to $U_\mathfrak{g} \to \text{End}(V)$.

**Poincare-Birkhoff-Witt Theorem:**
If $X_1, \ldots, X_N$ is a basis of $\mathfrak{g}$, then $i(X_1)^{n_1} \cdots i(X_N)^{n_N}$ form a basis of $U_\mathfrak{g}$.
(In particular $i: \mathfrak{g} \to U_\mathfrak{g}$ is injective.)

Proof: use induction on degree, skipped.

**Corollary:**
If $\mathfrak{h} \subset \mathfrak{g}$, then $U_\mathfrak{h} \subset U_\mathfrak{g}$.

**Verma module with highest weight $\mu$:**
Want to make $1 \in U_\mathfrak{g}$ to be the highest weight vector.
Take the left ideal $U_\mathfrak{g} \cdot \langle H - (\mu, H) \rangle$ for $H \in \mathfrak{h}, X_\alpha$ for $\alpha \in R^+$
(declaring 1 has weight $\mu$, and declaring 1 is a highest weight vector.)
Verma module is the quotient vector space
\[ W_\mu := U_\mathfrak{g}/U_\mathfrak{g} \cdot \langle H - (\mu, H) \rangle \text{ for } H \in \mathfrak{h}, X_\alpha \text{ for } \alpha \in R^+. \]
It loses the ring structure, but still has the left module structure of $U_\mathfrak{g}$. Thus it is a representation of $\mathfrak{g}$. Since $U_\mathfrak{g}$ is generated by $\mathfrak{g}$ (acting on 1), $[1] \in W_\mu$ is cyclic.

**Need:** $[1] \in W_\mu$ is non-zero. That is $1 \notin$ the left ideal in $U_\mathfrak{g}$.

**Proof:**
$g$ is complicated. Consider the following simpler subalgebras:

$$
\mathfrak{n}^\pm := \bigoplus_{\alpha \in \mathbb{R}^\pm} \mathfrak{g}_\alpha.
$$

$$
\mathfrak{b} := \mathfrak{n}^+ \oplus \mathfrak{h}.
$$

They are subalgebras since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha + \beta}$.

1 does not belong to the left ideal

$U_b \cdot (H - (\mu, H))$ for $H \in \mathfrak{h}$, positive root vectors $X$:

Consider the 1d representation of $b$: $(X + H) \cdot v = (\mu, H)v$.

(It is a representation since $[b, b] \subset \mathfrak{n}^+$ which acts by zero, and rank one representation is commutative. This is only valid for $b$ but not $g$.)

This corresponds to the morphism $U_b \to \mathbb{C}$. Kernel contains $U_b \cdot (H - (\mu, H), X_\alpha)$. $1_{U_b}$ maps to 1 and hence is not contained in kernel.

Now go back to $g = b \oplus \mathfrak{n}^-$. By PBW theorem, basis of $b$ and $\mathfrak{n}^-$ induce a basis of $U_g$. Hence $U_g = \bigoplus Y_{i_1} \ldots Y_{i_N} \cdot U_b$ where $\{Y_i : i = 1, \ldots, N\}$ is a basis for $\mathfrak{n}^-$. The ideal is $\bigoplus Y_{i_1} \ldots Y_{i_N} \cdot U_b \cdot H - (\mu, H) \bigoplus Y_{i_1} \ldots Y_{i_N} \cdot U_b \cdot X_\alpha$

We already know that $U_b \cdot (H - (\mu, H), X_\alpha)$ does not contain 1. Hence the above does not contain 1.

QED

For a basis $\{Y_i : i = 1, \ldots, N\}$ of $\mathfrak{n}^-$, $Y_1^{k_1} \ldots Y_N^{k_N} \cdot [1]$ form a basis of $W_\mu$:

It is obvious that they generate since $[1]$ is cyclic. They are linearly independent:

for a linear combination of $Y_1^{k_1} \ldots Y_N^{k_N}$ lying in the ideal of $U_b$, the coefficients must belong to both $\mathbb{C}$ and $U_b \cdot (H - (\mu, H), X_\alpha)$, which is just $\{0\}$ since $U_b \cdot (H - (\mu, H), X_\alpha) \nsubseteq 1$.

To make $W_\mu$ irreducible by quotient out certain invariant subspace.

Consider those vectors which can never get to $[1]$ by $X_\alpha$ action:

$$
U_\mu = \{v \in W_\mu : X_1 \cdot \ldots \cdot X_N \cdot v \text{ has no component in } [1] \text{ for any } X_i \in \mathfrak{n}^+\}.
$$

(Use weight space decomposition of $W_\mu$ to talk about the components.)

$U_\mu$ is invariant: $X_1 \cdot \ldots \cdot X_N \cdot Z \cdot v$ can be rearranged into $Y \ldots H \ldots X \ldots v$.

$X \ldots v$ has no component in $[1]$, and action by $Y \ldots H \ldots$ keeps this property.

Then take $V_\mu = W_\mu / U_\mu$.

Prop. $V_\mu$ is irreducible.

Proof:

Consider invariant subspace $S$ of $W_\mu$ containing but not equal to $U_\mu$.

Let $v \in S - U_\mu$. So $u = X_1 \cdot \ldots \cdot X_N \cdot v \in S$ has non-zero coefficient in $[1]$. Want to kill all other components of $u$.

Weight decomposition $u = a_0[1] + \sum_{\lambda \neq \mu} a_\lambda \cdot v_\lambda$. For $H \in \mathfrak{h}$,

$$
H \cdot u = a_0 (\mu, H) [1] + \sum_{\lambda \neq \mu} a_\lambda (\lambda, H) v_\lambda.
$$

Then $(H - (\lambda, H) \cdot 1d)$ kills the $\lambda$-component. The coefficient of $[1]$ becomes 0.
$a_0((\mu, H) - (\lambda, H))$, and we choose $H$ in the beginning to make sure $(\mu - \lambda, H) \neq 0$. Keep on doing this, $[1] \in S$, hence $S$ is the whole $W'_\mu$.

$V'_\mu$ is infinite-dimensional in general.

**Prop. For $\mu \in \tilde{C} \cap b^*_\mathfrak{z}, V'_\mu$ is finite-dimensional.**
(In geometry, positivity and integrality corresponds to whether the Kaehler class comes from ample line bundle (polarization).)

**Proof:**
The set of weights (which are integral) of $V'_\mu$ is invariant under Weyl group.
They are lower than the highest weight $\mu$, and hence finitely many.

Each weight space is finite dimensional:
Recall $Y^1 \cdots Y_N \cdot [1]$ form a basis of $W'_\mu$ for a basis $\{Y_i: i = 1, \ldots, N\}$ of $\mathfrak{z}^-$. Thus $Y^1 \cdots Y_N \cdot [1]$ spans the quotient $V'_\mu$.
There are just finitely many $Y^1 \cdots Y_N \cdot [1]$ with a given weight.

(If $\mu$ not integral, then don’t have Weyl symmetry. Look at $\mathfrak{sl}(2, \mathbb{C})$.
(Or consider $\mathfrak{s}_\alpha$-action where $\alpha \in \Delta$. We already know for $\mathfrak{sl}(2, \mathbb{C}), V'_\mu$ constructed in this way is finite-dimensional.)

Need to construct $W$-action by using action of $e^{\pi(X_\alpha)}, e^{\pi(Y_\alpha)}$ for $s_\alpha = \langle X_\alpha, Y_\alpha, H_\alpha \rangle \subset \mathfrak{g}$.
Since $V'_\mu$ may be infinite-dimensional, need to worry about exponential operators on $V'_\mu$. Once have $e^{\pi(X)}$, the same construction goes through for $W$-action. It is justified in the lemma below.
QED

**Def.** Locally nilpotent linear operator $X$ on $V$:
for all $v \in V, X^k v = 0$ for some $k > 0$.
(For finite dimensions, this is same as nilpotent. Just take a basis.)

**Have $e^X$ for locally nilpotent $X$.**

**Lemma:** For $s_\alpha = \langle X_\alpha, Y_\alpha, H_\alpha \rangle$ where $\alpha \in \Delta, X_\alpha, Y_\alpha$ act locally nilpotently on $V'_\mu$ if $\mu \in \tilde{C} \cap b^*_\mathfrak{z}$.

**Proof:** Sufficient to prove every $v \in V'_\mu$ is contained in a finite-dimensional sub-representation of $s_\alpha$.
The set $T$ of all such vectors form a vector space. It is invariant under whole $\mathfrak{g}$.
For $v \in T$, let $S \ni v$ be a finite-dimensional sub-representation of $s_\alpha$. Take $\mathfrak{g} \cdot S$ which is still finite-dimensional. Note that $\mathfrak{g} \cdot S$ is NOT invariant under $\mathfrak{g}$, but it is invariant under $s_\alpha$. $X \cdot v \in \mathfrak{g} \cdot S$ for any $X \in \mathfrak{g}$, and hence $X \cdot v \in T$.

$T \neq \emptyset$, and hence $T = V'_\mu$ as $V'_\mu$ is irreducible:
Take the highest weight vector $[1] \in V'_\mu$.
$(\mu, H_\alpha) \in \mathbb{Z}_{\geq 0}$ since $\mu \in \tilde{C} \cap b^*_\mathfrak{z}$ and $\alpha \in \Delta$. Then $\{Y^k \cdot [1]: k \in \mathbb{Z}\} \subset V'_\mu$ forms a finite-dimensional sub-representation of $s_\alpha$. Hence $[1] \in T$.

**Exercises. (Section 9.8)**
1. Show that the Verma module $W'_\mu$ is the maximal highest weight representation. Namely, for any highest weight representation $V'_\mu$, there is a surjective morphism $W'_\mu \to V'_\mu$ (and hence $V'_\mu$ is a quotient of $W'_\mu$).
2. Let $\mu \in \mathfrak{h}^* \cap R_+$ and $R_+ = \{\alpha_1, \ldots, \alpha_k\}$. Show that for the Verma module $W'_\mu$, the multiplicity of $\lambda$ is the number of $k$-tuples $(n_1, \ldots, n_k)$ such that $\lambda = \mu - n_1 \alpha_1 - \cdots - n_k \alpha_k$. 