Examples of Lie groups

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Motivation: want to describe a **smooth family of symmetries**.
ex. {rotations in the plane}. Robot arm.
More related to geometry: invertible changes that preserve a given geometric structure (ex. smooth structure, metric, complex structure…)

Lie group: has group structure and smooth manifold structure simultaneously.

Group structure:
Have multiplication, unit, and every element has an inverse.

Smooth structure:
smooth manifold such that multiplication and taking inverse are smooth.
(=> have **left and right multiplications as diffeomorphisms**)
Smooth manifold:
Have covering by charts to open sets in $\mathbb{R}^n$ such that transitions are smooth.

Why need smooth manifold?
ex $\mathbb{S}^1$ needs two charts to cover.

Examples: (Find tangent space at Identity)

1. $GL(n,\mathbb{R})$ (preserve real linear structure). $GL(n,\mathbb{C})$ (preserve linear complex structure; write it as a subgroup of $GL(n,\mathbb{R})$; Cauchy-Riemann equation).
2. (Closed subgroups of the above (called matrix Lie groups.)
3. $\mathbb{R}\times\mathbb{R}\times U(1)$ with multiplication defined by $(x_1+x_2, y_1+y_2, e^{i\,x_1 y_2}\,u_1\,u_2)$.
   NOT matrix group. Later: any $\mathbb{R}\times\mathbb{R}\times U(1)\to GL(n,\mathbb{C})$ has non-trivial kernel.
4. SL(n,ℝ) or SL(n,ℂ). Preserves oriented volume. $A^*(dV) = (\det A)\, dV$.
5. O(n). Preserves linear metric.
6. SO(n). Preserves linear metric and orientation.
7. $SO(3)\ltimes\mathbb{R}^3$ and products. Robot!
8. U(n). Preserves linear Hermitian metric. $U(n) = GL(2n,\mathbb{C})\cap O(2n)$.
   Reason: $h = g + i\omega$. $\omega(v,w) = g(I\cdot v, w)$.
9. $Sp(2n,\mathbb{R})$. Preserve linear symplectic structure. $U(n) = GL(2n,\mathbb{C})\cap Sp(2n,\mathbb{R})$.
10. SU(n). Preserves complex volume and Hermitian metric. "Calabi-Yau". "String".
11. $Sp(2n,\mathbb{C})$. Preserve linear complex symplectic structure.

$GL(2n,\mathbb{R}) \supset GL(2n,\mathbb{R})^+$; $GL(2n,\mathbb{R})^+ \subset GL(n,\mathbb{C})$ (complex) $\supset GL(n,\mathbb{R})$; $GL(n,\mathbb{C}) \supset U(n)$

$GL(2n,\mathbb{R})^+ \supset SL(2n,\mathbb{R}) \supset SO(2n) \supset U(n) \supset SU(n)$.
(Riemannian) (Kaehler) (Calabi-Yau)

$SL(2n,\mathbb{R}) \supset Sp(2n,\mathbb{R}) \supset U(n)$
(symplectic)

12. $Sp(n) = Sp(2n,\mathbb{C})\cap U(2n)$. Preserve linear hyperKaehler structure.

$GL(2n,\mathbb{C}) \supset GL(n,\mathbb{H}) \supset Sp(n)$

$GL(2n,\mathbb{C}) \supset SL(2n,\mathbb{C}) \supset SU(2n) \supset Sp(n)$ (hyper-Kaehler)

$SL(2n,\mathbb{C}) \supset Sp(2n,\mathbb{C}) \supset Sp(n)$
(complex symplectic)

13. O(3,1). Preserve linear Lorentz metric. Special relativity.
14. Heisenberg group
$$\left\{\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix} : a,b,c\in\mathbb{R}\right\}.$$

Linear hyperKaehler structure.
Consider $\mathbb{H}^n \cong \mathbb{C}^{2n}$ by $x + j\,y \mapsto (x,y)$; note that j is put in front of y so that right multiplication by i is not affected. It is considered as **right** $\mathbb{H}$-module.

$\mathbb{C}^n \simeq \mathbb{R}^{2n}$

$$\underbrace{(A+iB)}\cdot\overbrace{(\vec{c}+i\vec{d})}^{\binom{c}{d}} = \begin{pmatrix}A & \\ & A\end{pmatrix}\begin{pmatrix}c\\ d\end{pmatrix} + \overbrace{\begin{pmatrix}B & \\ & B\end{pmatrix}\begin{pmatrix} & -1\\ 1 & \end{pmatrix}}^{\begin{pmatrix} & -B\\ B & \end{pmatrix}}\begin{pmatrix}c\\ d\end{pmatrix}$$
$$= \begin{pmatrix}A & -B\\ B & A\end{pmatrix}\begin{pmatrix}c\\ d\end{pmatrix}.$$

$$2\cdot\partial_{\bar z} f = (\partial_x + i\,\partial_y)(f_R + i f_g) = 0.$$
$$\partial_x f_R = \partial_y f_g$$
$$\partial_y f_R = -\partial_x f_g$$

CR eq. $\Longleftrightarrow df = \begin{pmatrix}\partial_x f_R & \partial_y f_R\\ \partial_x f_g & \partial_y f_g\end{pmatrix} \in GL(n,\mathbb{C})$.

$$\left.\frac{d}{dt}\right|_{t=0}(\vec{a}_1(t)\wedge\ldots\wedge\vec{a}_n(t))$$
$$= \vec{a}_1'\wedge e_2\wedge\ldots\wedge e_n$$
$$+ \ldots$$
$$= (a_{11}' + a_{22}' + \ldots)\, dV$$
$$= \operatorname{tr} A'(0)\, dV.$$

- The underlying set is the Cartesian product $N\times H$.
- The group operation, $\bullet$, is determined by the homomorphism, φ:
  - $\bullet : (N\rtimes_\varphi H)\times(N\rtimes_\varphi H)\to N\rtimes_\varphi H$
    $(n_1,h_1)\bullet(n_2,h_2) = (n_1\varphi(h_1)(n_2),\, h_1h_2) = (n_1\varphi_{h_1}(n_2),\, h_1h_2)$

  for n_1, n_2 in N and h_1, h_2 in H.

$$h(v,w) = v^* w = (v_R^t - i v_g^t)(w_R + i w_g)$$
$$= \underbrace{(v_R^t w_R + v_g^t w_g)}_{g(v,w)} + i\underbrace{(v_R^t\cdot w_g - v_g^t\cdot w_R)}_{\omega(v,w)}$$

Linear hyperKaehler structure.

Consider $\mathbb{H}^n \cong \mathbb{C}^{2n}$ by $x + j\,y \mapsto (x, y)$; note that j is put in front of y so that right multiplication by i is not affected. It is considered as **right** $\mathbb{H}$-module.

$\mathbb{C}^{2n}$ has a standard complex structure I by entriwise multiplication by i.

Additional complex structure: $J \cdot (x, y) := (-\bar{y}, \bar{x})$. Coming from $(x + jy) \cdot j = -\bar{y} + j\bar{x}$.

$$j \cdot \overset{\mathbb{C}}{\overset{\cup\!\!\!\!\shortmid}{z}} = \bar{z} \cdot j.$$

Quaternionic linear map: $A \in GL(2n, \mathbb{C})$ with $A \cdot J = J \cdot \bar{A}$.

$$\begin{aligned}&(A + jB)(x + jy)\\ &= (Ax + j\bar{A}y) + (jBx - \bar{B}y)\\ &\sim \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\end{aligned}$$

(Note that the above equation is defined over $\mathbb{R}$. Thus $GL(n, \mathbb{H})$ is just a real Lie group, NOT a complex one.)

Also have standard holomorphic symplectic form $\omega_{\mathbb{C}}$ and standard Hermitian metric h (conjugate linear in first factor).

Define $K = I \circ J$. Then $I^2 = J^2 = K^2 = IJK = -Id$ (and so $IJ = K, JI = -K$ ...)

In other words we can treat it as a module over the quaternion algebra $\mathbb{H}$ (almost a field except noncommutative).

Have $\omega_{\mathbb{C}}(v, w) = h(Jv, w)$. Thus preserving $\omega_{\mathbb{C}}$ and h implies preserving J.

$$\begin{aligned}\left(J\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)^* \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &= \begin{pmatrix} -\bar{v}_2 \\ \bar{v}_1 \end{pmatrix}^* \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\\ &= -v_2^t w_1 + v_1^t \cdot w_2\\ &= \begin{pmatrix} v_1^t & v_2^t \end{pmatrix}\begin{pmatrix} & I \\ -I & \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.\end{aligned}$$

$Sp(n)$: quaternionic-linear and Hermitian.

$Sp(n) = Sp(2n, \mathbb{C}) \cap U(2n) = GL(n, \mathbb{H}) \cap U(2n) = GL(n, \mathbb{H}) \cap Sp(2n, \mathbb{C})$.

Story of quaternions (from Wikipedia)

In 1843, Hamilton knew that complex numbers could be viewed as points in a plane and that they could be added and multiplied together using certain geometric operations. Hamilton sought to find a way to do the same for points in space. Points in space can be represented by their coordinates, which are triples of numbers and have an obvious addition, but Hamilton had difficulty defining the appropriate multiplication.

According to a letter Hamilton wrote later to his son Archibald:

Every morning in the early part of October 1843, on my coming down to breakfast, your brother William Edward and yourself used to ask me: "Well, Papa, can you multiply triples?" Whereto I was always obliged to reply, with a sad shake of the head, "No, I can only add and subtract them."

On October 16, 1843, Hamilton and his wife took a walk along the Royal Canal in Dublin. While they walked across Brougham Bridge (now Broom Bridge), a solution suddenly occurred to him. While he could not "multiply triples", he saw a way to do so for *quadruples*. By using three of the numbers in the quadruple as the points of a coordinate in space, Hamilton could represent points in space by his new system of numbers. He then carved the basic rules for multiplication into the bridge:

$i^2 = j^2 = k^2 = ijk = -1.$

From <https://en.wikipedia.org/wiki/History_of_quaternions>

Note: in the above we mostly concern with linear structures. We easily get to `infinite-dimensional Lie groups' if we consider geometric structures on a manifold.
ex. Diffeo(M), Sympl(M),...
The matrix groups can be treated as structure groups of the tangent space.
ex. real manifold, complex manifold, symplectic manifold, Riemannian manifold, Kaehler manifold, Calabi-Yau manifold, hyperKaehler manifold...
WARNING: we have skipped INTEGRABILITY CONDITIONS on geometric structures.
Riemannian manifolds are classified by their holonomy groups.

Exercises. (Section 1.6)

3. Sp(2,$\mathbb{R}$)=SL(2,$\mathbb{R}$);
   Sp(2,$\mathbb{C}$)=SL(2,$\mathbb{C}$);
   Sp(1)=SU(2).
4. Let a be irrational and G consists of $\begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix}$
   for $t \in \mathbb{R}$.
   Show that the closure in $GL(2, \mathbb{C})$ gives the torus consisting of $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$

for $\theta, \phi \in \mathbb{R}$.