Closed subgroup theorem
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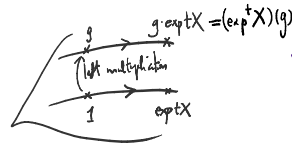
Ref: Lee - Introduction to Smooth Manifolds Ch. 19, 20.

Lie(G) := $T_1G \cong$ {left-G-invariant vector fields}.
Integrating along $X \in Lie(G)$ gets a
one-parameter subgroup of diffeomorphisms $\exp^t X = \exp tX : G \to G$.
$(\exp(s+t)X = \exp sX \circ \exp tX.)$
$X \in \mathrm{Lie}(G)$ is complete: walk a bit, and left multiplication by the endpoint.
Abuse of notation: under the diffeomorphism $\exp tX$, denote $1 \mapsto \exp tX \in G$.
This defines **exp**: $\mathfrak{g} \to G$. (Don't even use metric!)
Since X is left-invariant, the diffeomorphism $\exp^t X$ is $g \mapsto g \exp tX$ (right multiplication.)
$d\exp|_0 = \mathrm{Id}$. Hence exp is a local diffeomorphism.



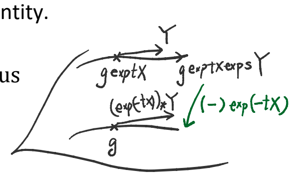
Def: $\begin{cases} \frac{d}{dt}\left((\exp^t X)(g)\right) = X|_{(\exp^t X)(g)} \\ (\exp^0 X)(g) = g \end{cases}$

Denote $(\exp^t X)(1) = \exp^t X$.
$(\exp^t X)(g) = g \cdot \exp^t(X)$:
$\begin{cases} \frac{d}{dt}\left(g \cdot \exp^t(X)\right) = g \cdot \left(X|_{\exp^t X}\right) = X|_{g \cdot \exp^t X} \\ g \exp^0(X) = g. \end{cases}$

Denote $\exp^1 X = \exp X$.
$\exp tX = \exp^t X$:
$\exp^s tX = \exp^{ts} X$ since both satisfy the same D.E.
- $\frac{d}{ds} \exp^s tX = tX|_{\exp^s tX}$
- $\frac{d}{ds} \exp^{ts} X = t \cdot X|_{\exp^{ts} X}$

$\exp^{s+t} X = \exp^s X \exp^t X$:
$\begin{cases} \frac{d}{dt} \exp^s X \exp^t X = (\exp^s X)_* X|_{\exp^t X} = X|_{\exp^s X \exp^t X}. \\ \exp^s X \exp^t X|_{t=0} = \exp^s X \end{cases}$
$\exp^{s+t} X$ satisfies the same D.E.

Lie bracket on Lie(G): $[X,Y] = L_X Y$ (Lie derivative).
Lie derivative satisfies $[X,Y] = -[Y,X]$ and Jacobi identity.
$L_X Y$ is still left-invariant:
$L_X Y|_g = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} g \exp tX \exp sY \exp(-tX)$. Thus
$h \cdot (L_X Y|_g) = L_X Y|_{hg}$.

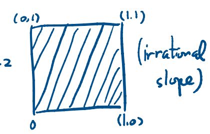


$L_X Y|_g = \frac{d}{dt}(\exp -tX)_* Y|_{g \cdot \exp tX}$.

Lie(G) is a **Lie algebra**: a vector space with $[-,-]$.
Lie subalgebra: subspace closed under $[-,-]$.

Lie subgroup H: Image of an injective group homomorphism to G which is an **immersion**. (NOTE: NEED NOT EMBEDDED, that is, may not be homeomorphism to image!)

ex. $\mathbb{R} \hookrightarrow T^2$



Theorem:
Subalgebra of Lie(G) <-> connected Lie subgroup of G.

<- is trivial: take tangent space at 1. Closed under Lie bracket since H is a submanifold near 1.
-> follows from the **Frobenius theorem** in differential topology:
A sub-bundle of TM whose sheaf of local sections are closed under $[-,-]$ integrates to a foliation.
Then take the leaf containing 1. It is closed under multiplication:
suppose $g, h \in \mathrm{leaf}_1$. $g \cdot h = L_g \cdot h \in L_g \cdot \mathrm{leaf}_1 = \mathrm{leaf}_g = \mathrm{leaf}_1$.

(**Foliation**: a collection of disjoint connected immersed submanifolds (called leafs) whose union is the whole space M, and can take local coordinates $x_1, \ldots, x_n$ of M such that the leafs are given by taking $x_{k+1}, \ldots, x_n$ to be constants.)

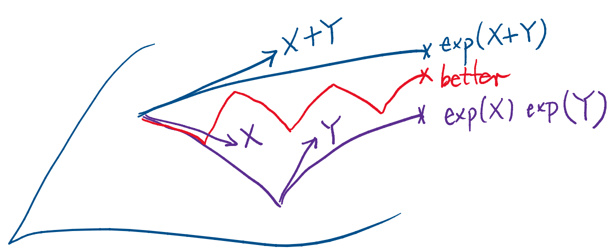
Closed subgroup theorem:
A closed subgroup H of G must be a submanifold (that is embedded).
(Note: don't need H to be Lie subgroup in the condition.)

topology + group str.
$\Rightarrow$ smooth str.

Proof:
Need to restricts charts of G to charts of H at all $h \in H$. Consider an open set U of $\mathfrak{g}$ where exp is a diffeomorphism.
Want to argue $h \cdot \exp$ restricted to $U \cap \mathfrak{h}$ provides a chart of H. Need to define $\mathfrak{h}$!
$\mathfrak{h} := \{X \in \mathfrak{g} : \exp tX \in H \text{ for all } t \in \mathbb{R}\}$.
$\mathfrak{h} \subset \mathfrak{g}$ is a vector subspace: $0 \in \mathfrak{h}$. Closed under scaling.
$X + Y \in \mathfrak{h}$ if X and Y are: consider $\exp t(X+Y)$.
TRICKY: $\exp t(X+Y) \neq (\exp tX)(\exp tY)$!
For t small,
$(\exp tX)(\exp tY) = \exp(\phi(t))$ for some smooth path ϕ. $(\phi(0) = 0.)$
Take $\frac{d}{dt}\Big|_{t=0}$, get $\phi'(0) = X + Y$. Thus $\phi = t(X+Y) + t^2 Z(t)$.
Replace t by $\frac{t}{n}$:
$\left(\left(\exp \frac{tX}{n}\right)\left(\exp \frac{tY}{n}\right)\right)^n = \exp\left(t(X+Y) + \frac{t^2}{n} Z\left(\frac{t}{n}\right)\right)$ and hence

$$\lim_{n\to\infty} \left(\left(\exp \frac{tX}{n}\right)\left(\exp \frac{tY}{n}\right)\right)^n = \exp t(X+Y)$$

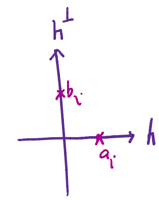
Since H is closed, LHS belongs to H.
Thus $\mathfrak{h} \subset \mathfrak{g}$ is a vector subspace.

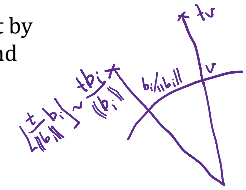


Need to take U sufficiently small such that $\exp(U \cap \mathfrak{h}) = (\exp U) \cap H$. (Always have $\exp(U \cap \mathfrak{h}) \subset (\exp U) \cap H$.) Then $h \cdot \exp$ restricted to $U \cap \mathfrak{h}$ provides a chart of H around h.

$\mathfrak{h}^\perp$
$\uparrow$

Need to take U sufficiently small such that $\exp(U \cap \mathfrak{h}) = (\exp U) \cap H$. (Always have $\exp(U \cap \mathfrak{h}) \subset (\exp U) \cap H$.) Then $h \cdot \exp$ restricted to $U \cap \mathfrak{h}$ provides a chart of H around h.
Assume such U does not exist. Then have a sequence of points in H converging to 1 but not in $\exp(U \cap \mathfrak{h})$.
Take a meric on $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$.
Then the points can be written as $(\exp a_i)(\exp b_i)$ for $a_i \in \mathfrak{h}$ and $b_i \in \mathfrak{h}^\perp$ (since $(\exp a)(\exp b)\colon \mathfrak{h} \times \mathfrak{h}^\perp \to G$ is local diffeo).
NOTE THAT $\exp b_i \in H$ since $(\exp a_i)(\exp b_i) \in H$. But still $\exp b_i/m$ may not in H for some $m \in \mathbb{Z}$.
b_i are normalized to points on the unit sphere. Take a convergent subsequence and denote its limit by $v \in \mathfrak{h}^\perp$. For any $t \in \mathbb{R}$, tv is a limit of $\{t_i b_i\}$ for some $t_i \in \mathbb{Z}_{>0}$. (This uses $|b_i| \to 0$.) $\exp t_i b_i \in H$, and hence $\exp tv \in H$ since H is closed! Then $v \in \mathfrak{h}$ by definition of $\mathfrak{h}$, a contradiction!





Lie homomorphism $G \to H$:
smooth group homomorphism.

Theorem:
Continuous homomorphism $G \to H$ is automatically smooth!

Proof:
First, any continuous homomorphism $\gamma\colon \mathbb{R} \to H$ is $\gamma(t) = \exp tX$ for some X (and hence smooth):
For t_0 small, $\gamma(t_0) = \exp t_0 X$ for some X. $\gamma(t_0) = \gamma\left(\frac{t_0}{k}\right)^k = \exp\left(\frac{t_0 X}{k}\right)^k$ and hence
$\gamma\left(\frac{t_0}{k}\right) = \exp\left(\frac{t_0 X}{k}\right)$. (In the region that exp is diffeomorphism, $(-)^k \in G$ corresponds to $k \cdot (-) \in \mathfrak{g}$ which is injective.)
Then $\gamma\left(\frac{p\, t_0}{k}\right) = \exp\left(\frac{p\, t_0 X}{k}\right)$ for all p, k. By continuity $\gamma(t) = \exp tX$.
Now consider $\Phi\colon G \to H$, $1_G \mapsto 1_H$. Use charts provided by exp to understand the map.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\Phi"'] & H \end{tikzcd}$$

$\Phi \circ \exp_G(X) = \exp_H \circ\, \phi(X)$. ϕ is a priori only defined near $X = 0$.
For each X, $\Phi \circ \exp_G(tX)$ gives a continuous homomorphism $\mathbb{R} \to H$. From above $\Phi \circ \exp_G(tX) = \exp_H tY$. Such Y is unique since $\exp_H$ is a local diffeomorphism.
We define $\phi(X) = Y$. Thus $\phi\colon \mathfrak{g} \to \mathfrak{h}$.
Suffice to prove ϕ is linear, and hence Φ is smooth around 1_G.
Then $\Phi = L_{\Phi(g)} \circ \Phi \circ L_{g^{-1}}$ (since it is homomorphism) is smooth around g.

ϕ is linear:
$\Phi \circ \exp_G(t\, sX) = \exp_H\, t\, s\phi(X)$. $\phi(sX)$ also satisfies this. By uniqueness $\phi(sX) = s\phi(X)$.

$$\exp_H t\,\phi(X+Y) = \Phi \circ \exp_G t(X+Y) = \Phi\left(\lim_{n\to\infty}\left(\exp_G\left(\frac{tX}{n}\right)\exp_G\left(\frac{tY}{n}\right)\right)^n\right)$$
$$= \lim_{n\to\infty} \Phi\left(\exp_G\left(\frac{tX}{n}\right)\exp_G\left(\frac{tY}{n}\right)\right)^n \text{ (}\Phi\text{ is continuous)}$$
$$= \lim_{n\to\infty}\left(\left(\Phi \circ \exp_G \frac{tX}{n}\right)\left(\Phi \circ \exp_G \frac{tY}{n}\right)\right)^n \text{ (}\Phi\text{ is homomorphism)}$$
$$= \lim_{n\to\infty}\left(\left(\exp_H \phi\left(\frac{tX}{n}\right)\right)\exp_H \phi\left(\frac{tY}{n}\right)\right)^n$$
$$= \lim_{n\to\infty}\left(\left(\exp_H \frac{t\,\phi(X)}{n}\right)\left(\exp_H \frac{t\,\phi(Y)}{n}\right)\right)^n$$
$$= \exp_H t(\phi(X) + \phi(Y)).$$

By uniqueness $\phi(X+Y) = \phi(X) + \phi(Y)$.

Exercises. (Section 2.6)

5. Show that
$\exp\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} = \begin{pmatrix} e^a & b\frac{e^a - e^d}{a-d} \\ 0 & e^d\end{pmatrix}$ (where the right hand side is defined by taking limit when $a = d$.
8. Consider $X = \begin{pmatrix} 1 & 2 \\ -2 & 1\end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 2 \\ 2 & 1\end{pmatrix}$. Compute e^{tX} and e^{tY} by diagonalization. Visualize the curves $e^{tX} \cdot v$ and $e^{tY} \cdot v$ for $v \neq 0$.