\[ \begin{align*}
\mathfrak{gl}(2n, \mathbb{R}) & \to \mathfrak{sl}(2n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{u}(n) \to \mathfrak{su}(n) \\
& \to \mathfrak{sp}(2n, \mathbb{R})
\end{align*} \]

**Lie(G):**
1. \(\mathfrak{gl}(2n, \mathbb{R})\): all matrices.
2. \(\mathfrak{sl}(2n, \mathbb{R})\): \(\text{tr}(X) = 0\). Obtained by taking \(\frac{d}{dt}
\bigg|_{t=0} \frac{d}{ds} \exp tX \exp sY \exp(-tX)\) on \(\det(A(t)) = 1\) where \(A(0) = \text{Id}\) and \(\frac{d}{dt}
\bigg|_{t=0} A(t) = X\).
3. \(\mathfrak{so}(2n, \mathbb{R})\): \(X = -X^T\).
4. \(\mathfrak{gl}(2n, \mathbb{C})\): all complex matrices.
5. \(\mathfrak{sp}(2n, \mathbb{C})\):
\[\begin{pmatrix}
0 & -\text{Id} \\
\text{Id} & 0
\end{pmatrix}\bigg)^T = \begin{pmatrix}
0 & -\text{Id} \\
\text{Id} & 0
\end{pmatrix}.
\]
6. \(u(n)\): \(X = -X^*\).
7. \(\mathfrak{su}(n)\): \(X = -X^*\) and \(\text{tr}(X) = 0\).

**Lie derivative:**
\[ad(X) \cdot Y = [X, Y] = \frac{d}{dt} \bigg|_{t=0} \frac{d}{ds} \exp tX \exp sY \exp(-tX)\]
For \(\mathfrak{gl}(2n, \mathbb{R})\):
\[XY - YX\].

**Adjoint action** on \(g = \text{Lie}(G) = T_eG\) in general: \(\text{Ad}(g) \cdot Y = R_{g^{-1}} \cdot Y\big|_g\).
For \(\mathfrak{gl}(2n, \mathbb{R})\):
\[gYg^{-1}\].

\[ad(X) = \frac{d}{dt}
\bigg|_{t=0} \text{Ad}(\exp tX)\].

\(\text{Ad}(g)\) can be understood as right multiplication by \(g^{-1}\) on (left-invariant) vector fields. Denote \(\text{Ad}(g) \cdot Y = Y \cdot g^{-1}\). Obvious that
\[\text{Ad}(g \cdot h) = \text{Ad}(g) \cdot \text{Ad}(h)\].

\[\text{Ad}(g) \cdot [X, Y] = [\text{Ad}(g) \cdot X, \text{Ad}(g) \cdot Y];
\]
\[\frac{d}{dt} \bigg|_{t=0} (Y \exp -tX) \cdot g^{-1} = \frac{d}{dt} \bigg|_{t=0} (Y \cdot g^{-1}) \cdot (g \cdot (\exp -tX) \cdot g^{-1})
\]
\[= \frac{d}{dt} \bigg|_{t=0} (Y \cdot g^{-1}) \cdot (\exp(-tX \cdot g^{-1})) = [\text{Ad}(g) \cdot X, \text{Ad}(g) \cdot Y]\]

**Killing form** in general: \(\langle X_1, X_2 \rangle = \text{tr}(ad(X_1) \circ ad(X_2)\) on \(g)\).
For \(\mathfrak{gl}(n, \mathbb{R})\): \(2n \text{tr}(X_1X_2) - 2 \text{tr}(X_1) \text{tr}(X_2)\).
Thus it is \(2n\) \(\text{tr}(X_1X_2)\) for \(\mathfrak{sl}(n, \mathbb{R})\) which is positive definite on symmetric matrices and negative definite on skew-symmetric matrices (and hence indefinite on \(\mathfrak{sl}(n, \mathbb{C})\)).
Can verify by using the basis \(e_{ij}\) of \(\mathfrak{gl}(n, \mathbb{R})\).
\(\langle X \cdot X \rangle = n \cdot \text{tr}(X) = \text{tr}(X); \text{tr}(X_1 \cdot (-) \cdot X_2) = (\text{tr}(X_1) \text{tr}(X_2)\).
For \(\mathfrak{su}(n)\): \(2n\) \(\text{tr}(X_1X_2)\). Verify in a similar way. This is negative definite since \(X_2 = -X_2\).

**Adjoint action always preserve the Killing form:**
\[\text{tr}(ad(X_1) \circ ad(X_2)) = \text{tr}(\text{Ad}(g) \cdot X_1) \circ ad(\text{Ad}(g) \cdot X_2))\]
\[\text{ad}(\text{Ad}(g) \cdot X_1) \circ ad(\text{Ad}(g) \cdot X_2); Y = [\text{Ad}(g) \cdot X_1, \text{Ad}(g) \cdot X_2, Y] = \text{Ad}(g)[X_1, X_2, \text{Ad}(g^{-1}) \cdot Y]\]
\[= \text{Ad}(g) \cdot ad(X_1) \circ ad(X_2) \circ ad(\text{Ad}(g)^{-1}\]
whose trace equals to \(\text{tr}(ad(X_1) \circ ad(X_2))\).
Examples of Lie homomorphisms:
1. det: $GL(n, \mathbb{C}) \to \mathbb{C}^\times$. Ker $= SL(n, \mathbb{C})$.
2. $\mathbb{R} \to SO(2)$ rotation by $\theta \in \mathbb{R}$.
3. $Ad: G \to GL(g)$. Correspondingly $ad: g \to gl(g)$ is a Lie algebra homomorphism.
   Moreover $Ad(g): g \to g$ is a Lie algebra homomorphism.
4. $SU(2) \to SO(3)$ by acting on $su(2)$ (space of skew-Hermitian matrices) by $gXg^{-1}$. The adjoint action preserves the Killing form which is just the standard metric (up to scaling).
   
   $SU(2) \cong S^3$: Identify $\alpha + j \beta \in \mathbb{H}$ as $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$.
   Then $SU(2) = \{[\alpha + \beta j] : |\alpha + \beta j|^2 = 1\} \subset \mathbb{H}$.
   Conjugate transpose is quaternionic conjugation.
   Matrix multiplication is quaternionic multiplication.
   The homomorphism is $2:1$. Consider preimage of $Id$: $uxu^{-1} = x$ for all $x \in \text{Im} \mathbb{H}$. Then $u \in \mathbb{R}$. But $|u|^2 = 1$, and hence $u = \pm 1$. Indeed $\pm u$ maps to the same rotation in $SO(3)$. Hence $SO(3) = S^3/\pm = \mathbb{RP}^3$.
   Note that $ux\bar{u}$ fixes $u$ and $\bar{u}$, and hence fixes $\frac{u-\bar{u}}{2} \in \text{Im} \mathbb{H}$. This spans the axis of rotation. Normalize $\frac{u-\bar{u}}{2}$ to $h$.
   Then $u = \cos \frac{\theta}{2} + h \sin \frac{\theta}{2}$ for some $\theta$. Then $ux\bar{u}$ is rotation by $\theta$. For instance take $h = i$, then $ux\bar{u} = e^{i\theta/2} (ai + bj + ck) e^{-i\theta/2} = ai + e^{i\theta} (bj + ck)$ which is rotating the $\{j,k\}$-plane by $\theta$.
   Thus the homomorphism is surjective.
5. Given a Lie homomorphism $\Phi: G \to H$, have the tangent map $\Phi: g \to \mathfrak{h}$ with $\Phi \circ \exp_G(X) = \exp_H \circ \Phi(X)$.
   Consider $\Phi: SU(2) \to SO(3), g \mapsto Ad(g)$. $\Phi(X) = ad(X) = [X, -]$.
   Explicit: take the basis $E_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, E_2 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, E_3 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $su(2)$. Then $[E_1, E_2] = E_3$, $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$.
   Thus $ad(E_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, ad(E_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, ad(E_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Exercises. (Section 3.9)
10. Show that there is a linear isomorphism $\Phi: su(2) \to \mathbb{R}^3$ such that $\Phi([X,Y]) = \Phi(X) \times \Phi(Y)$ (the cross product for $\mathbb{R}^3$).
11. Show that $su(2)$ and $sl(2, \mathbb{R})$ are not isomorphic Lie algebras.