Adjoint actions

Tuesday, February 6, 2018 9:30 AM

$$\mathfrak{gl}(2n,\mathbb{R}) \supset \mathfrak{sl}(2n,\mathbb{R}) \begin{matrix}\supset \mathfrak{so}(2n,\mathbb{R}) \supset\\ \supset \mathfrak{gl}(n,\mathbb{C}) \supset\\ \supset \mathfrak{sp}(2n,\mathbb{R}) \supset\end{matrix} \mathfrak{u}(n) \supset \mathfrak{su}(n)$$

**Lie(G)**:

1. $\mathfrak{gl}(2n,\mathbb{R})$: all matrices.
2. $\mathfrak{sl}(2n,\mathbb{R})$: $\mathrm{tr}(X)=0$. Obtained by taking $\frac{d}{dt}\big|_{t=0}$ on $\det A(t)=1$ where $A(0)=\mathrm{Id}$ and $\frac{d}{dt}\big|_{t=0} A(t)=X$.
3. $\mathfrak{so}(2n,\mathbb{R})$: $X=-X^T$.
4. $\mathfrak{gl}(2n,\mathbb{C})$: all complex matrices.
5. $\mathfrak{sp}(2n,\mathbb{C})$:
$$\left(\begin{pmatrix}0 & -Id\\ Id & 0\end{pmatrix}X\right)^T=\begin{pmatrix}0 & -Id\\ Id & 0\end{pmatrix}X.$$
6. $\mathfrak{u}(n)$: $X=-X^*$.
7. $\mathfrak{su}(n)$: $X=-X^*$ and $\mathrm{tr}(X)=0$.

**Lie derivative**:

$$ad(X)\cdot Y=[X,Y]=\frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0}\exp tX \exp sY \exp(-tX)$$

For $\mathfrak{gl}(2n,\mathbb{R})$: $XY-YX$.

**Adjoint action** on $\mathfrak{g}=\mathrm{Lie}(G)=T_1G$ in general: $Ad(g)\cdot Y=R_{g^{-1}}\cdot Y|_g$.

For $\mathfrak{gl}(2n,\mathbb{R})$: $gYg^{-1}$.

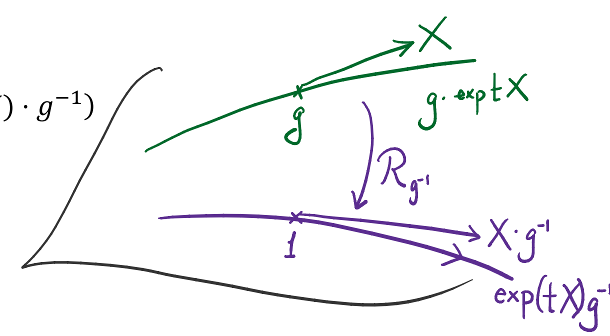
$ad(X)=\frac{d}{dt}\big|_{t=0} Ad(\exp tX)$.

$Ad(g)$ can be understood as right multiplication by $g^{-1}$ on (left-invariant) vector fields. Denote $Ad(g)\cdot Y=Y\cdot g^{-1}$. Obvious that $Ad(g\cdot h)=Ad(g)\circ Ad(h)$.

$Ad(g)\cdot[X,Y]=[Ad(g)\cdot X, Ad(g)\cdot Y]$:

$$Ad(g)\cdot[X,Y]=\frac{d}{dt}\bigg|_{t=0}(Y\cdot\exp -tX)\cdot g^{-1}=\frac{d}{dt}\bigg|_{t=0}(Y\cdot g^{-1})\cdot(g\cdot(\exp -tX)\cdot g^{-1})$$

$$=\frac{d}{dt}\bigg|_{t=0}(Y\cdot g^{-1})\cdot(\exp(-t\,X\cdot g^{-1}))=[Ad(g)\cdot X, Ad(g)\cdot Y]$$




**Killing form** in general: $\langle X_1,X_2\rangle=\mathrm{tr}(ad(X_1)\circ ad(X_2)$ on $\mathfrak{g})$.

For $\mathfrak{gl}(n,\mathbb{R})$: $2n\,\mathrm{tr}(X_1X_2)-2\,\mathrm{tr}(X_1)\mathrm{tr}(X_2)$.

Thus it is $2n\,\mathrm{tr}(X_1X_2)$ for $\mathfrak{sl}(n,\mathbb{R})$ which is positive definite on symmetric matrices and negative definite on skew-symmetric matrices (and hence indefinite on $\mathfrak{sl}(n,\mathbb{R})$).

Can verify by using the basis $e_{ij}$ of $\mathfrak{gl}(n,\mathbb{R})$.

$\mathrm{tr}(X\,\cdot)=n\,\mathrm{tr}(X)=\mathrm{tr}(\cdot\,X)$; $\mathrm{tr}(X_1\cdot(-)\cdot X_2)=(\mathrm{tr}\,X_1)(\mathrm{tr}\,X_2)$.

For $\mathfrak{su}(n)$: $2n\,\mathrm{tr}(X_1X_2)$. Verify in a similar way. This is negative definite since $X_2=-X_2^*$.

**Adjoint action always preserve the Killing form**:

$\mathrm{tr}(ad(X_1)\circ ad(X_2))=\mathrm{tr}(ad(Ad(g)\cdot X_1)\circ ad(Ad(g)\cdot X_2))$.

$$ad(Ad(g)\cdot X_1)\circ ad(Ad(g)\cdot X_2)\cdot Y=\big[Ad(g)\cdot X_1,[Ad(g)\cdot X_2,Y]\big]=Ad(g)\big[X_1,[X_2,Ad(g^{-1})\cdot Y]\big]$$

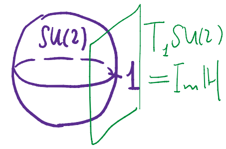
$$=Ad(g)\circ ad(X_1)\circ ad(X_2)\circ Ad(g)^{-1}$$

whose trace equals to $\mathrm{tr}(ad(X_1)\circ ad(X_2))$.

**Examples of Lie homomorphisms**:

1. det: $GL(n,\mathbb{C}) \to \mathbb{C}^{\times}$. Ker $= SL(n,\mathbb{C})$.
2. $\mathbb{R} \to \mathrm{SO}(2)$ rotation by $\theta \in \mathbb{R}$.
3. $Ad: G \to GL(\mathfrak{g})$. Correspondingly $ad: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism.
   Moreover $Ad(g): \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism.
4. $\mathrm{SU}(2) \to \mathrm{SO}(3)$ by acting on $\mathfrak{su}(2)$ (space of skew-Hermitian matrices) by $gXg^{-1}$. The adjoint action preserves the Killing form which is just the standard metric (up to scaling).
   $SU(2) \cong \mathbb{S}^3$: Identify $\alpha + j\,\beta \in \mathbb{H}$ as $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$. Then $SU(2) = \{|\alpha + \beta j|^2 = 1\} \subset \mathbb{H}$.

   $(A+jB)(x+jy)$
   $= (Ax + j\bar{A}y) + (jBx - \bar{B}y)$
   $\sim \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

   Conjugate transpose is quaternionic conjugation.
   Matrix multiplication is quaternionic multiplication.
   $\mathfrak{su}(2)$ is identified as Im $\mathbb{H}$. ($\alpha \in i\mathbb{R}$ and hence skew-Herm. & tr=1.)
   The adjoint action is $uxu^{-1} = ux\bar{u}$.
   The homomorphism is 2:1. Consider preimage of Id: $uxu^{-1} = x$ for all $x \in$ Im $\mathbb{H}$. Then $u \in \mathbb{R}$. But $|u|^2 = 1$, and hence $u = \pm 1$. Indeed $\pm u$ maps to the same rotation in $SO(3)$. Hence $SO(3) = \mathbb{S}^3/\pm = \mathbb{RP}^3$.
   Note that $ux\bar{u}$ fixes $u$ and $\bar{u}$, and hence fixes $\frac{u-\bar{u}}{2} \in$ Im $\mathbb{H}$. This spans the axis of rotation. Normalize $\frac{u-\bar{u}}{2}$ to $h$.
   Then $u = \cos\theta/2 + h\sin\theta/2$ for some $\theta$. Then $ux\bar{u}$ is rotation by $\theta$. For instance take $h = i$, then
   $ux\bar{u} = e^{i\theta/2}\,(ai + bj + ck)\,e^{-i\theta/2} = ai + e^{i\theta}\,(bj + ck)$ which is rotating the {j,k}-plane by $\theta$.
   Thus the homomorphism is surjective.
5. Given a Lie homomorphism $\Phi: G \to H$, have the tangent map $\phi: \mathfrak{g} \to \mathfrak{h}$ with $\Phi \circ \exp_G(X) = \exp_H \circ\, \phi(X)$.
   Consider $\Phi: \mathrm{SU}(2) \to \mathrm{SO}(3)$, $g \mapsto Ad(g)$. $\phi(X) = ad(X) = [X, -]$.
   Explicit: take the basis $E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, E_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, E_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\mathfrak{su}(2)$. Then
   $[E_1, E_2] = E_3, [E_2, E_3] = E_1, [E_3, E_1] = E_2$. Thus
   $ad(E_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, ad(E_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, ad(E_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

**Exercises. (Section 3.9)**

10. Show that there is a linear isomorphism $\phi: \mathfrak{su}(2) \to \mathbb{R}^3$ such that $\phi([X,Y]) = \phi(X) \times \phi(Y)$ (the cross product for $\mathbb{R}^3$).
11. Show that $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$ are not isomorphic Lie algebras.