Matrix exponential

Have exp : g(\(n, \mathbb{C}\)) → GL(\(n, \mathbb{C}\)). The defining property is
\[
\frac{d}{dt}\exp tX = X|_{\exp tX} = (L(\exp tX))_0 = (\exp tX) \cdot X \text{ with } \exp 0 = 1_{GL(n)}.
\]
(Since GL(\(n\)) is an open subset of Mat_{\(n\times n\)}, tangent space at any point is identified with Mat_{\(n\times n\)})

\[
e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}
\]
satisfies the defining properties.

Indeed it absolutely converges under the standard complete Hilbert-Schmidt matrix norm.
(Note that |XY| ≤ |X| · |Y|. Apply Cauchy-Schwartz on each entry.)

From the definition,
\begin{enumerate}
  \item \(e^{X^*} = e^X^*\).
  \item \(e^{X+Y} = e^X e^Y = e^Y e^X \text{ if } XY = YX\).
  \item \(|e^X| ≤ e^{||X||}\).
  \item \(e^{CX C^{-1}} = Ce^XC^{-1}\).
\end{enumerate}

To compute \(e^X\), take Jordan canonical form \(S + N\) where \(S\) is diagonal, \(N\) is nilpotent.
Then \(e^{C(S+N)C^{-1}} = Ce^SC^{-1}\).

exp is invertible in a small neighborhood of 0 and hence has inverse. Want to have
\[
\log A = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}(A - I)^m}{m} \text{ for } A \in GL(n) \text{ to be its inverse.}
\]

The above series is complex analytic in the disc \(|z - 1| < 1\).
\(e^{\log z} = z\) holds since it holds on the interval (0,2) (identity theorem).
To talk about \(\log e^w = w\), need \(|w - 1| < 1\).
\[
|e^w - 1| = \left|w + \frac{w^2}{2!} + \ldots\right| ≤ |w| + \left|\frac{|w|^2}{2!} + \ldots\right| = e^{|w|} - 1 < 1
\]
if \(|w| < \log 2\). Then by identity theorem \(\log e^w = w\).

Now consider \(|A - I| < 1\). Then the above series converges.
For diagonal matrix \(A\), it is obvious that \(e^{\log A} = A\). Since every matrix is the limit of diagonalizable matrices, this still holds for general matrices.
(Consider Jordan form. Then perturb diagonal. Have distinct eigenvalues.)
Similarly \(\log e^X = X\), if \(|e^X - I| < 1\). This is ensured by \(|X| < \log 2\).

**Theorem:**
\(\exp : g(\(n, \mathbb{C}\)) → GL(\(n, \mathbb{C}\))\) is surjective.

**Proof:**
Take Jordan canonical form. For each block \(J = \lambda I + N = \lambda \left( I + \frac{N}{\lambda} \right)\) consider
\[
\log J = (\log \lambda) I + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \left( \frac{N}{\lambda} \right)^m}{m}
\]
which makes sense since \((N)^m = 0\) for \(m\) large enough.
Then \(\exp \log J = J\): cannot use the previous argument since \(\frac{N}{\lambda}\) can be big (and after perturbing to diagonalizable matrices, the series may no longer converge).

Consider \(J_t = \lambda \left( I + \frac{t N}{\lambda} \right)\) by previous result \(\exp \log J_t = J_t\) for \(t\) small. Also both LHS and RHS are analytic in \(t\) (indeed log is now polynomial). Hence true for all \(t\).

**Note that \(\exp \) is NOT injective:** ex. circle. \(\log \lambda\) has different branches. So exp is invertible only in a small neighborhood (so that we can consistently take the branch
log 1 = 0.

Note: $\exp: \mathfrak{gl}(n, \mathbb{R}) \to GL(n, \mathbb{R})$ is NOT surjective. (Nor $\mathfrak{sl}(n, \mathbb{R}) \to SL(n, \mathbb{R})$.)

For instance take $\begin{pmatrix} -4 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$. Its square root have at least four eigenvalues $2i, -2i$,

\[ \frac{i}{2}, -\frac{i}{2} \] (since the char. poly. for the square root, a real matrix, is invariant under conjugation).

Note: For compact Riemannian manifold, exp is surjective.

Even when $\exp$ is not surjective, every $g$ can be written as $\exp X_1 \ldots \exp X_k$ for some $X_1, \ldots, X_k$ if $G$ is connected:
The set of all such elements is both open and closed: exp is a local diffeomorphism on $U$ around 0.
For $\exp X_1 \ldots \exp X_k$, we have an open neighborhood $\exp X_1 \ldots \exp X_k \cdot \exp U$ inside the set. Hence it is open.
If $g_i \to g$ where $g_i$ are such elements, $g_i \in g \cdot \exp U$ for $i$ big enough. $g^{-1}g_i = \exp X_i$. Hence $g = g_i \exp -X = \exp X_1 \ldots \exp X_k \exp -X$. Thus it is closed.

In particular the image of the exponential map may not be a subgroup: the image generates the whole connected Lie group, but some elements may not lie in the image.

Also $\exp$ may not be an open map (even for compact Lie group): consider $SU(2)$.
$\begin{pmatrix} \pi i & 0 \\ 0 & -\pi i \end{pmatrix} \in su(2)$ is mapped to -1. Open neighborhood: two distinct eigenvalues (near $\pi i$ and $-\pi i$) with eigen-directions close to (1,0) and (0,1).
Image under exp: same eigen-directions, exp of the eigenvalues.

But near $-1 \in SU(2)$, have $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with eigenvectors (1,i) and (1,-i). They are not contained in the image and hence image is not open.
(The problem is that exp maps the whole unit sphere in $su(2)$ to a point.)

**Exercises. (Section 2.6)**

4. Using the Jordan canonical form, show that every complex square matrix is the limit of a sequence of diagonalizable matrices. (For instance consider the vectors $e_1 + \epsilon e_2 + \cdots + \epsilon^{k-1} e_k$.)
10. Using $\log \infty$, show that for $X \in \mathfrak{gl}(n, \mathbb{C})$,

\[
\lim_{m \to \infty} \left( 1 + \frac{X}{m} \right)^m = e^X.
\]