Matrix exponential

Tuesday, January 30, 2018 11:00 AM

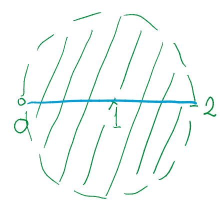
Have $\exp : \mathfrak{gl}(n,\mathbb{C}) \to GL(n,\mathbb{C})$. The defining property is
$\frac{d}{dt}\exp tX = X\Big|_{\exp tX} = (L_{(\exp tX)})_* X\Big|_0 = (\exp tX)\cdot X$ with $\exp 0 = 1_{GL(n)}$.
(Since $GL(n)$ is an open subset of $\mathrm{Mat}_{n\times n}$, tangent space at any point is identified with $\mathrm{Mat}_{n\times n}$.)

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$$
satisfies the defining properties.
Indeed it absolutely converges under the standard complete **Hilbert-Schmidt matrix norm**.
(Note that $|XY| \le |X|\cdot|Y|$. Apply Cauchy-Schwartz on each entry.)

From the definition,
1. $(e^X)^* = e^{X^*}$.
2. $e^{X+Y} = e^X e^Y = e^Y e^X$ if $XY = YX$.
3. $|e^X| \le e^{|X|}$.
4. $e^{CXC^{-1}} = Ce^X C^{-1}$.

To **compute** e^X, take Jordan canonical form $S+N$ where S is diagonal, N is nilpotent.
Then $e^{C(S+N)C^{-1}} = Ce^S e^N C^{-1}$.

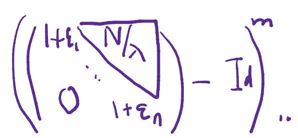
exp is invertible in a small neighborhood of 0 and hence has inverse. Want to have
$$\log A = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}(A-I)^m}{m}$$ for $A \in GL(n)$ to be its inverse.



The above series is **complex analytic** in the disc $|z-1|<1$.
$e^{\log z} = z$ holds since it holds on the interval $(0,2)$ (identity theorem).
To talk about $\log e^w = w$, need $|e^w - 1| < 1$.
$$|e^w - 1| = \left|w + \frac{w^2}{2!} + \cdots\right| \le |w| + \frac{|w|^2}{2!} + \cdots = e^{|w|} - 1 < 1$$
if $|w| < \log 2$. Then by identity theorem $\log e^w = w$.

Now consider $\{|A - I| < 1\}$. Then the above series converges.
For diagonal matrix A, it is obvious that $e^{\log A} = A$. Since **every matrix is the limit of diagonalizable matrices**, this still holds for general matrices.
(Consider Jordan form. Then perturb diagonal. Have distinct eigenvalues.)
Similarly $\log e^X = X$, if $|e^X - I| < 1$. This is ensured by $|X| < \log 2$.

Theorem:
$\exp : \mathfrak{gl}(n,\mathbb{C}) \to GL(n,\mathbb{C})$ is surjective.
Proof:
Take Jordan canonical form. For each block $J = \lambda I + N = \lambda\left(I + \frac{N}{\lambda}\right)$ consider
$$\log J := (\log\lambda) I + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\left(\frac{N}{\lambda}\right)^m}{m}$$



which makes sense since $(N)^m = 0$ for m large enough.
Then $\exp\log J = J$: cannot use the previous argument since $\left|\frac{N}{\lambda}\right|$ can be big (and *after perturbing to diagonalizable matrices, the series may no longer converge*).
Consider $J_t = \lambda\left(I + \frac{tN}{\lambda}\right)$. By previous result $\exp\log J_t = J_t$ for t small. Also both LHS and RHS are analytic in t (indeed log is now polynomial). Hence true for all t.

Note that exp is NOT injective: ex. circle. $\log\lambda$ has different branches. So exp is invertible only in a small neighborhood (so that we can consistently take the branch

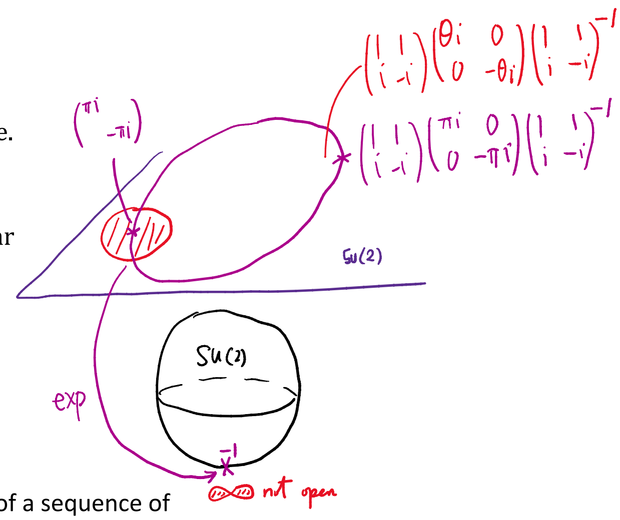
$\log 1 = 0$.)

Note: $\mathbf{exp}: \mathfrak{gl}(\boldsymbol{n}, \mathbb{R}) \to \boldsymbol{GL}(\boldsymbol{n}, \mathbb{R})$ **is NOT surjective**. (Nor $\mathfrak{sl}(n, \mathbb{R}) \to SL(n, \mathbb{R})$.)
For instance take $\begin{pmatrix} -4 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$. Its square root have at least four eigenvalues $2i, -2i$, $\frac{i}{2}, -\frac{i}{2}$ (since the char. poly. for the square root, a real matrix, is invariant under conjagation).

Note: For compact Riemannian manifold, exp is surjective.

Even when exp is not surjective,
every g can be written as $\exp X_1 \dots \exp X_k$ for some $X_1, \dots, X_k$ if G is connected:
The set of all such elements is both open and closed: exp is a local diffeomorphism on U around 0.
For $\exp X_1 \dots \exp X_k$, we have an open neighborhood $\exp X_1 \dots \exp X_k \cdot \exp U$ inside the set. Hence it is open.
If $g_i \to g$ where g_i are such elements, $g_i \in g \cdot \exp U$ for i big enough. $g^{-1} g_i = \exp X$.
Hence $g = g_i \exp -X = \exp X_1 \dots \exp X_k \exp -X$. Thus it is closed.

In particular the **image of the exponential map may not be a subgroup**: the image generates the whole connected Lie group, but some elements may not lie in the image.



Also **exp may not be an open map** (even for compact Lie group): consider $SU(2)$.
$\begin{pmatrix} \pi i & 0 \\ 0 & -\pi i \end{pmatrix} \in \mathfrak{su}(2)$ is mapped to -1. Open neighborhood: two distinct eigenvalues (near πi and $-\pi i$) with eigen-directions close to (1,0) and (0,1).
Image under exp: same eigen-directions, exp of the eigenvalues.
But near $-1 \in SU(2)$, have $-\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with eigenvectors (1,i) and (1,-i). They are not contained in the image and hence image is not open.
(The problem is that exp maps the whole unit sphere in $\mathfrak{su}(2)$ to a point.)

Exercises. (Section 2.6)

4. Using the Jordan canonical form, show that every complex square matrix is the limit of a sequence of diagonalizable matrices. (For instance consider the vectors $e_1 + \epsilon\, e_2 + \cdots + \epsilon^{k-1} e_k$.)
10. Using log ⬚, show that for $X \in \mathfrak{gl}(n, \mathbb{C})$,
$$\lim_{m\to\infty} \left(1 + \frac{X}{m}\right)^m = e^X.$$