Baker-Campbell-Hausdorff formula

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Want to compare e^{X+Y} and $e^X e^Y$ more precisely.
Recall: $e^{tX}e^{tY} = e^{t(X+Y)+o(t^2)}$ for small tX and tY.
May also write as: $\log(e^{tX}e^{tY}) = t(X+Y) + o(t^2)$.

The formula: (for X, Y small enough)

$$\log e^X e^Y = X + \int_0^1 g\left(e^{ad_X} e^{t\, ad_Y}\right) Y\, dt$$

$$= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\big[X,[X,Y]\big] - \frac{1}{12}\big[Y,[X,Y]\big] + \cdots$$

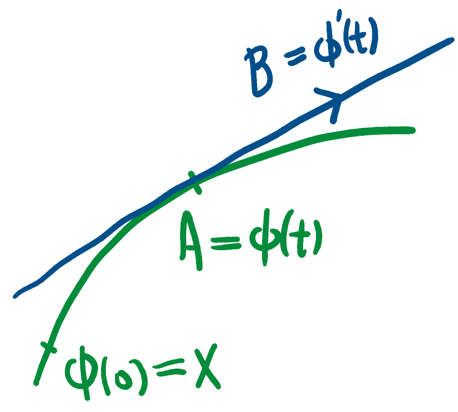
where $g(z)$ is the series expression for

$$\frac{\log z}{1 - z^{-1}} = \sum a_m (z-1)^m .$$

Note: can expand $\log e^X e^Y$ and compute terms-by-terms. But hard to get close formula in this way.
Local group structure around Id is determined by Lie bracket.

Proof:
Think about $e^{\phi(t)} = e^X e^{tY}$. Want to find

$$\phi(1) = \phi(0) + \int_0^1 \phi'(t)\, dt = X + \int_0^1 \phi'(t)\, dt .$$



To compute $\phi'(t)$, differentiate on both sides:

$$(d\exp \square)\Big|_{\phi(t)} \phi'(t) = e^X e^{tY} \cdot Y.$$

LHS equals to

$$\frac{d}{ds}\Big|_{s=0} e^{A+sB}$$

where $A = \phi(t), B = \phi'(t)$.

Note that it is linear on B.

Key Lemma:

$$\left.\frac{d}{ds}\right|_{s=0} e^{A+sB} = e^{A} \cdot \frac{Id - \exp(-ad_A)}{ad_A} \cdot B$$

for any $A, B \in \mathfrak{g}$.

RHS makes sense since $\frac{1-e^{-z}}{z}$ is an entire function.

Assuming this, plug this to LHS,

$$e^{\phi(t)} \cdot \frac{Id - \exp(-ad_{\phi(t)})}{ad_{\phi(t)}} \cdot \phi'(t) = e^{X} e^{tY} \cdot Y.$$

By def. $e^{\phi(t)} = e^{X} e^{tY}$. Hence

$$\frac{Id - \exp(-ad_{\phi(t)})}{ad_{\phi(t)}} \cdot \phi'(t) = Y.$$

For X and t small, LHS is close to Id and hence invertible.

$$\phi'(t) = \left(\frac{Id - \exp(-ad_{\phi(t)})}{ad_{\phi(t)}}\right)^{-1} \cdot Y = \left(\frac{Id - (Ad_{e^{\phi(t)}})^{-1}}{\log Ad_{e^{\phi(t)}}}\right)^{-1} \cdot Y$$

$$= \left(\frac{Id - (Ad_{e^X} \circ Ad_{e^{tY}})^{-1}}{\log(Ad_{e^X} \circ Ad_{e^{tY}})}\right)^{-1} \cdot Y = \frac{\log(Ad_{e^X} \circ Ad_{e^{tY}})}{Id - (Ad_{e^X} \circ Ad_{e^{tY}})^{-1}} \cdot Y$$

$$= \frac{\log(e^{ad_X} \circ e^{t\, ad_Y})}{Id - (e^{ad_X} \circ e^{t\, ad_Y})^{-1}} \cdot Y.$$

(Note that the operators commute and so we can write as fraction.)

Both sides are analytic in t, and hence true for all t.

Proof of key lemma:

If A were small, then $\left.\frac{d}{ds}\right|_{s=0} e^{A+sB} \sim B$.

Consider

$$\left.\frac{d}{ds}\right|_{s=0} e^{A+sB} = \left.\frac{d}{ds}\right|_{s=0} \left(e^{\frac{A+sB}{m}}\right)^m = \sum_{k=0}^{m-1} e^{\frac{(m-k-1)A}{m}} \cdot \left(\left.\frac{d}{ds}\right|_{s=0} e^{\frac{A+sB}{m}}\right) \cdot e^{\frac{kA}{m}}$$

$$= e^{\frac{(m-1)A}{m}} \sum_{k=0}^{m-1} Ad_{e^{\frac{-kA}{m}}} \left(\left.\frac{d}{ds}\right|_{s=0} e^{\frac{A+sB}{m}}\right)$$

$$= e^{\frac{(m-1)A}{m}} \sum_{k=0}^{m-1} \exp\left(ad_{\frac{-kA}{m}}\right) \left(\left.\frac{d}{ds}\right|_{s=0} e^{\frac{A+sB}{m}}\right)$$

$$= e^{\frac{(m-1)A}{m}} \sum_{k=0}^{m-1} \exp\left(\frac{-k}{m}\ ad_A\right) \left(\left.\frac{d}{ds}\right|_{s=0} e^{\frac{A}{m}+s\frac{B}{m}}\right)$$

$$= e^{\frac{(m-1)A}{m}} \left(\frac{1}{m} \cdot \sum_{k=0}^{m-1} \exp\left(\frac{-k}{m}\ ad_A\right)\right) \left(\left.\frac{d}{ds}\right|_{s=0} e^{\frac{A}{m}+sB}\right)$$

where $\left.\frac{d}{ds}\right|_{s=0} e^{\frac{A}{m}+s\frac{B}{m}} = \frac{1}{m} \cdot \left.\frac{d}{ds}\right|_{s=0} e^{\frac{A}{m}+sB}$ by chain rule.

$\frac{1}{m} \cdot \sum_{k=0}^{m-1} \exp\left(\frac{-k}{m}\ ad_A\right)$ is the Riemann sum of

$\int_0^1 \exp(-t\ ad_A)\ dt$.

$$\int_0^1 \exp(-tz)\ dt = \left[\frac{\exp(-tz)}{-z}\right]_0^1 = \frac{\exp(-z) - 1}{-z} = \frac{1 - e^{-z}}{z}$$

where both sides are entire function in z.

This still holds for matrix: it is true for diagonalizable matrices, which form a dense set.

Thus taking $m \to \infty$ gives

$$\left.\frac{d}{ds}\right|_{s=0} e^{A+sB} = e^A \cdot \frac{Id - e^{-ad_A}}{ad_A} \cdot B.$$

$$\left(1+\frac{1}{z-1}\right)\cdot\left(\sum_{k>0}\frac{(-1)^{k-1}(z-1)^k}{k}\right)$$

$$=\sum_{k>0}\frac{(-1)^{k-1}(z-1)^k}{k}+\sum_{k>0}\frac{(-1)^{k-1}(z-1)^{k-1}}{k}$$

$$=1+\sum_{k>0}\left(\frac{(-1)^{k-1}}{k}+\frac{(-1)^k}{k+1}\right)(z-1)^k$$

$$=1+\sum_{k>0}\frac{(-1)^{k-1}}{k(k+1)}(z-1)^k.$$

$$e^{ad_X}e^{t\,ad_Y}=\sum_{\ell,p\geq 0}\frac{ad_X^\ell}{\ell!}\frac{t^p\,ad_Y^p}{p!}$$

Leading term:

$$1+\frac{1}{2}\left(\sum_{\ell,p\geq 0}\frac{ad_X^\ell}{\ell!}\frac{t^p\,ad_Y^p}{p!}-Id\right)-\frac{1}{6}\left(\sum_{\ell,p\geq 0}\frac{ad_X^\ell}{\ell!}\frac{t^p\,ad_Y^p}{p!}-Id\right)^2+\ldots\ldots$$

$$ad_X+t\,ad_Y+\frac{ad_X^2}{2}+t\,ad_X\,ad_Y+\frac{t^2}{2}ad_Y^2+\ldots\ldots$$

$$=1+\frac{1}{2}\left(ad_X+t\,ad_Y+\frac{ad_X^2}{2}+t\,ad_X\,ad_Y+\frac{t^2}{2}ad_Y^2\right)$$

$$-\frac{1}{6}\left(ad_X^2+t(ad_X\,ad_Y+ad_Y\,ad_X)+t^2\,ad_Y^2\right)+\ldots$$

$$\int_0^1=1+\frac{1}{2}\left(ad_X+\frac{1}{2}ad_Y+\frac{ad_X^2}{2}+\frac{1}{2}ad_X\,ad_Y+\frac{1}{6}ad_Y^2\right)$$

$$-\frac{1}{6}\left(ad_X^2+\frac{1}{2}(ad_X\,ad_Y+ad_Y\,ad_X)+\frac{1}{3}ad_Y^2\right)+\ldots$$

$$\log e^Xe^Y=X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\ldots\ldots$$