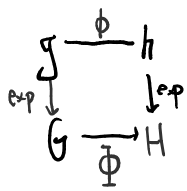
Lie correspondence

Wednesday, February 14, 2018 8:27 AM

Recall that a Lie group homomorphism $\Phi: G \to H$ has tangent map $\phi: \mathfrak{g} \to \mathfrak{h}$.

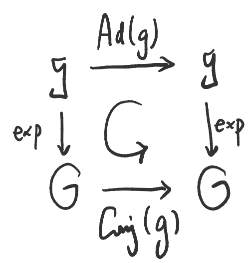


Since $\Phi(gxg^{-1}) = \Phi(g)\Phi(x)\Phi(g)^{-1}$, $\phi \circ \mathrm{Ad}_G(g) = \mathrm{Ad}_H\big(\Phi(g)\big) \circ \phi$.
Then $\phi \circ \mathrm{ad}_{\mathfrak{g}}(X) = \mathrm{ad}_{\mathfrak{h}}\big(\phi(X)\big) \circ \phi$, that is $\phi([X,Y]) = [\phi(X), \phi(Y)]$.
Thus **Lie group homomorphism corresponds to a Lie algebra homomorphism**.

In particular take $\Phi = \mathrm{Ad}\colon\ G \to GL(\mathfrak{g})$. It corresponds to the Lie algebra homomorphism $\phi = \mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.
Hence

$$\mathbf{Ad}(\mathbf{exp}\,\boldsymbol{X}) = \mathbf{exp}(\mathbf{ad}\,\boldsymbol{X}) = \sum_{n=0}^{\infty} \frac{(\mathrm{ad\ X})^n}{n!}.$$

Thus **$\exp X$ commutes with $\exp tY$ if $[X, Y] = 0$** using that $\exp \circ \mathrm{Ad(g)} = \mathrm{Conj(g)} \circ \exp$ (because conjugation pushes a left invariant vector field to a left invariant one).



Take $G = GL(n, \mathbb{C})$. Then

$$\boldsymbol{e^X\, Y\, e^{-X}} = \sum_{n=0}^{\infty} \frac{(\mathbf{ad\ X})^{n} \cdot \boldsymbol{Y}}{\boldsymbol{n}!} = \sum_{n=0}^{\infty} \frac{\big[\boldsymbol{X}, \big[\boldsymbol{X}, \dots, [\boldsymbol{X}, \boldsymbol{Y}]\big]\big]}{\boldsymbol{n}!}.$$

If G is connected, then group homo $\boldsymbol{\Phi}$ is determined by its induced alg homo $\boldsymbol{\phi}$:
Recall that any element $g = e^{X_1} \cdot \dots \cdot e^{X_n}$ if G is connected. $\Phi(g) = e^{\phi(X_1)} \cdot \dots \cdot e^{\phi(X_n)}$.
In particular **$\mathfrak{g}$ commutative implies (connected) G commutative**. (ad=Id, which is the tangent map for both Ad and Id.)

Any real Lie algebra (whose underlying vector space is over $\mathbb{R}$) can be complexified. For instance, $\mathfrak{gl}(n, \mathbb{R}) \otimes \mathbb{C} \cong \mathfrak{gl}(n, \mathbb{C}) \cong \mathfrak{u}(n) \otimes \mathbb{C}$.

Recall: have correspondence between connected Lie subgroups and Lie subalgebras (Frobenius theorem). (Don't need G connected for this.)

Thm:
Suppose G is connected. There exists a one-to-one correspondence between
connected normal Lie subgroups and ideals of Lie(G).

Recall: H is normal means $GHG^{-1} = H$.
Lie subgroup means a subgroup which is also an immersed submanifold.

Proof:
=>) For $X \in \mathfrak{g}, Y \in \mathfrak{h}$, $\exp tX \exp sY \exp -tX \in H$. Taking derivatives gives $[X, Y] \in \mathfrak{h}$.

<=) $Ad(e^X)\cdot Y = \sum_{n=0}^{\infty} \frac{(\text{ad X})^n \cdot Y}{n!} \in \mathfrak{h}$.
Since $\exp \circ \text{Ad}(e^X) = \text{Conj}(e^X) \circ \exp$, $e^X \cdot e^Y \cdot e^{-X} \in H$.
Since H is connected, any $h = e^{Y_1} \cdot \ldots \cdot e^{Y_n}$. Hence $e^X \cdot h \cdot e^{-X} \in H$.
Since G is connected, any $g = e^{X_1} \cdot \ldots \cdot e^{X_n}$. Hence $g \cdot h \cdot g^{-1} \in H$.

Lemma:
If $G \to H$ induces an isomorphism $\mathfrak{g} \to \mathfrak{h}$ and H is simply connected, then $G \to H$ is an isomorphism:
$G \to H$ induces an isomorphism on the tangent spaces together with left multiplications implies $G \to H$ must be a covering map (inverse image of each neighborhood U is disjoint union of sheets homeo. to U).
Reason: kernel is closed subgroup and hence a submanifold. Local iso. implies kernel consists of discrete points.
Consider $Id \in V \subset G$ with $V \xrightarrow{\cong} U \subset H$. Can shrink V such that it no longer intersects with $g \cdot V \ \forall g \in Ker$.
If H is simply connected, then $G \to H$ must be a diffeomorphism.

Thm:
If G is simply connected, then any $\mathfrak{g} \to \mathfrak{h}$ integrates to $G \to H$.

Proof:
Consider the graph of $\mathfrak{g} \to \mathfrak{h}$ which is a subalgebra in $\mathfrak{g} \oplus \mathfrak{h}$. It corresponds to a Lie subgroup Gr in $G \times H$.
The first projection $Gr \to G$ is an isomorphism and hence invertible:
the corresponding map $\mathfrak{gr} \to \mathfrak{g}$ is an isomorphism, and G is simply connected.
Thus have $G \to Gr$. Compose it with the second projection $Gr \to H$ to get $G \to H$.

Conclusion:

Lie algebra	<->	Simply connected Lie group	
$X \in \mathfrak{g}$		all $g = e^{X_1} \ldots e^{X_n} \in G$	if G is connected
Lie subalgebra	<->	connected Lie subgroup (which can be immersed)	
ideal	<->	connected normal Lie subgroup	if G is connected
$\exp(\text{ad}(X))$	<->	$\text{Ad}(\exp X)$	
$[\mathfrak{g}, \mathfrak{g}] = 0$	<->	gh=hg	if G is connected
$d_{Id}\Phi_1 = d_{Id}\Phi_2$	<->	$\Phi_1 = \Phi_2$	if G is connected
$d_{Id}\Phi$ is iso.	<->	Φ: G → H is iso	if H is simply connected
algebra homomorphism	<->	group homomorphism Φ: G → H	if G is simply connected

Thm:
Every Lie algebra can be written as $Lie(G)$ for some G, which is unique if G is simply connected.

This follows from
Ado's theorem:
Every Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(N, \mathbb{C})$.

Thus $\mathfrak{g}$ corresponds to $G \subset GL(N, \mathbb{C})$. Take universal cover. (Note that the universal cover may no longer be a matrix group.)
If there is another simply connected G' with $Lie(G') = \mathfrak{g}$, Id on $\mathfrak{g}$ induces $G \to G'$, which is iso. by above.

Exercises. (Section 3.9)
1. Give an example of $G \subset GL(n, \mathbb{C})$ and $X \in \mathfrak{gl}(n, \mathbb{C})$ such that $e^X \in G$ but $X \notin \text{Lie}(G)$.
2. Let $A \in SL(2, \mathbb{R})$ which has an eigenvalue in $\mathbb{C} - \mathbb{R}$. Show that

$$A = C \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} C^{-1}$$

for some invertible real matrix C.