Nilpotent and solvable ideals

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Idea: quotient out ideal to get smaller Lie algebra.
Try to get quotient which captures the "most non-trivial part of Lie bracket".
**ex.** $\mathfrak{g} = \{\text{matrices with last } (n-k) \text{ rows being zero}\}$.
$\mathfrak{g} = \mathfrak{sl}(k) \oplus c \cdot I_k \oplus \mathfrak{n}$.

**Semi-simple:** does not contain any non-zero Abelian ideal.
**Simple**: $\dim \mathfrak{g} \geq 2$ and the only ideals are $\{0\}$ and itself.
Have classification by Dynkin diagram.
It turns out:
Semi-simple = direct sum of simples.

**For instance, $\mathfrak{sl}(2, \mathbb{C})$ is simple: take the basis**
$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
$[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H$.
Suppose $Z = aX + bY + cH$ is in the ideal. $[X, Z] = bH - 2cX$. $[X, [X, Z]] = -2bX$. If $b \neq 0$, then $X$ is in the ideal. Then $[X, Y] = H, [H, Y] = -2Y$ imply that H and Y are also in the ideal which has to be $\mathfrak{sl}(2, \mathbb{C})$.
The case $b = 0$: do the same argument for $[Y, [Y, Z]]$, then the ideal is $\mathfrak{sl}(2, \mathbb{C})$ unless $a = 0$.
The case $a = b = 0$: $[H, X] = 2X, [H, Y] = -2Y$ implies the ideal is $\mathfrak{sl}(2, \mathbb{C})$ unless $c = 0$. The ideal is *{0}* if $a = b = c = 0$.

**Commutator ideal**: $[\mathfrak{g}, \mathfrak{g}] = \text{Span}\,\{[X, Y]: X, Y \in \mathfrak{g}\}$.

An ideal is in particular a Lie subalgebra.
Keep on taking commutator ideals, get $\mathfrak{g}_0 = \mathfrak{g},\ \mathfrak{g}_{j+1} = [\mathfrak{g}_j, \mathfrak{g}_j] \subset \mathfrak{g}_j, \ldots$ called **derived series**. All these are ideals.
**Solvable**: $\mathfrak{g}_j = \{0\}$ for some j.
(If $\mathfrak{g}$ is simple, $\mathfrak{g}_j = \mathfrak{g}$ for all j.)

**Levi decomposition**: any Lie algebra is the semi-direct product of a solvable ideal and a semi-simple subalgebra.
Semi-direct product $\mathfrak{g} \oplus \mathfrak{a}$: fix $\mathfrak{g} \to \text{Der}(\mathfrak{a})$

$[(x,s),(y,t)] = ([x,y],[s,t] + x\cdot t - y\cdot s)$.

Similar concept: $\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^{j+1} = [\mathfrak{g},\mathfrak{g}^j] = \left[\mathfrak{g},\left[\mathfrak{g},[\dots,\mathfrak{g}]\right]\right] \subset \mathfrak{g}^j$. Sequence of ideals called **lower central series**.
**Nilpotent**: $\mathfrak{g}^j = \{0\}$ for some j.
$\mathfrak{g}_j \subset \mathfrak{g}^j$. Hence **nilpotent implies solvable**.

For instance, the Lie algebra of nilpotent upper triangular matrices $\mathfrak{n}$ is nilpotent (and hence solvable).
Can take the basis $E_{i,j}, j > i$. Then
$[E_{i,j}, E_{k,l}] = 0$ if $j \neq k$ and $= E_{i,l}$ if $j = k$.
$\mathfrak{n}^k$ is spanned by $E_{i,j}$ for $j - i > k$ and hence $\mathfrak{n}^{n-1} = \{0\}$.

The Lie algebra of upper triangular matrices $\mathfrak{u}$ is solvable but not nilpotent.
Can take the basis $E_{i,i}, E_{i,i+1}, E_{i,i+2}, \dots, E_{1,n}$. Similar as above and $[E_{i,i}, E_{j,j}] = 0$.
$\mathfrak{u}_1 = \mathfrak{u}^1 = \mathfrak{n}$. So $\mathfrak{u}_n = 0$. But $\mathfrak{u}^i = \mathfrak{u}^1$ for all $i \geq 1$.

**Lemma.** If I is solvable ideal of $\mathfrak{g}$ and $\mathfrak{g}/I$ is solvable, then $\mathfrak{g}$ is solvable.
**Proof:** $\mathfrak{g}_j \subset I$ for big j since $\mathfrak{g}/I$ solvable. Since $I$ solvable, $\mathfrak{g}_j$ is solvable, and hence $\mathfrak{g}$ is solvable.

Solvable radical $SR$:
the largest solvable ideal.
**Prop.** Sum of solvable ideals is still solvable, and hence $SR$ exists and is unique.
**Proof:** $(I+J)/J = I/(I\cap J)$. RHS is solvable. $J$ is solvable. By lemma $I+J$ is solvable.

Have
$0 \to SR \to \mathfrak{g} \to \mathfrak{g}/SR \to 0$.
$\mathfrak{g}/SR$ is semi-simple: if it has an Abel. ideal $I$, then preimage $\hat{I}$ in $\mathfrak{g}$ is a bigger solvable ideal. ($[\hat{I},\hat{I}] \subset SR$.)
Levi: this sequence splits as Lie algebras, that is, can invert the last arrow. (Note: always split as vector spaces. But need the inverted arrow being Lie homo.)

**Proof of Levi decomposition**:
Try to split $\mathfrak{g}$. Once we have $\mathfrak{g} \cong SR \oplus \mathfrak{g}_0$ for Lie subalgebra $\mathfrak{g}_0$, then $\mathfrak{g}_0 \cong \mathfrak{g}/SR$ which is semi-simple, done.

Consider the semi-simple $\mathfrak{g}/SR$. If adjoint action on $\mathfrak{g}$ can descend to $\mathfrak{g}/SR$, then $\mathfrak{g}$ (as a module of $\mathfrak{g}/SR$) splits into the submodule $SR$ and a complementary submodule $\mathfrak{g}_0$, which is an ideal and hence a subalgebra. Done.

Otherwise: $[SR, \mathfrak{g}] \neq 0$.

**Reduce to the case that $\mathfrak{g}$ has no non-zero ideal properly contained in SR by taking quotient:**
Suppose it is already true in such a case. Do induction on $\dim \mathfrak{g}$.
$\dim \mathfrak{g} = 1$: $I = SR = \mathfrak{g}$. Trivial.
Now for general $\mathfrak{g}$, if it has a non-zero ideal $I$ properly contained in $SR$,
$\mathfrak{g}/I$ has lower dim. whose SR is $SR/I$.
Inductive assumption gives $\mathfrak{g}/I = SR/I \oplus \mathfrak{g}_0/I$ (as v.s.).
Then $\mathfrak{g} = SR + \mathfrak{g}_0$. However it is not a direct sum.
Since $\mathfrak{g}_0/I$ is semi-simple, $I$ is the SR of $\mathfrak{g}_0$.
$\mathfrak{g}_0$ has lower dimension, and by inductive assumption $\mathfrak{g}_0 = I \oplus \mathfrak{g}_0'$ where $\mathfrak{g}_0'$ is semi-simple.
Hence $\mathfrak{g} = SR + \mathfrak{g}_0'$. $SR \cap \mathfrak{g}_0' \subset \mathfrak{g}_0'$ must be zero since $\mathfrak{g}_0'$ is semi-simple.
Thus $\mathfrak{g} = SR \oplus \mathfrak{g}_0'$.

Now **for $\mathfrak{g}$ *with no non-zero ideal properly contained in SR***:
$[SR, SR] = 0$: $[SR, SR] \neq SR$ since the derived series goes to zero.

**Key idea: vector space splitting corresponds to projection to $SR$.**
Consider $\phi \in End(\mathfrak{g}), \phi(\mathfrak{g}) \subset SR$ with $\phi|_{SR} = c \cdot Id$ for some c.
Mainly interested in $c = 1$. Consider general $c$ to form *a* $\mathfrak{g}$*-module* $C$.
$End(\mathfrak{g}) = \mathfrak{g}^* \otimes \mathfrak{g}$ is a $\mathfrak{g}$-module by product rule:
$(X \cdot \phi)(Y) = [X, \phi(Y)] - \phi([X, Y])$. $(*)$
$(X \cdot \phi)(\mathfrak{g}) \subset SR$. $(X \cdot \phi)|_{SR} = 0$.
Thus
$\mathfrak{g} \cdot C \subset B$ where $B \subset C$ is the submodule consisting of
$\phi \in End(\mathfrak{g}), \phi(\mathfrak{g}) \subset SR$ with $\phi|_{SR} = 0$.

The actual projections that we are interested ($c = 1$) lie in complement of $B$ in $C$.
Want to use semi-simple $\mathfrak{g}/SR$ to split the module.
Need: they are modules of $\mathfrak{g}/SR$.
$SR \cdot B = 0$ since $[SR, SR] = 0$ (Consider (*).)
But $SR \cdot C \neq 0$:
$(X \cdot \phi)(-) = -c_\phi \cdot [X, -]$ for $X \in SR$.
Thus need to mod out
$A \coloneqq \{ad_X : X \in SR\} \subset B$.
$A$ is also a $\mathfrak{g}$-submodule:
$(Z \cdot ad_X)(Y) = [Z, ad_X(Y)] - ad_X([Z, Y]) = \big[Y, [Z, X]\big] = ad_{-[Z,X]}(Y)$.
Then have $B/A$ and $C/A$ as $\mathfrak{g}/SR$-mod.
$\mathfrak{g}/SR$ is semi-simple. Then there exists a complementary module $D/A$:
$B/A \bigoplus D/A = C/A$.
Thus $B + D = C$ and $B \cap D = A$.
Take a projection $\phi \in D$ with $\phi|_{SR} = Id$.

Note: $X \cdot \phi$ induces a projection of $X \in \mathfrak{g}$ to $SR$:
Recall $\mathfrak{g} \cdot C \subset B$, and hence $\mathfrak{g} \cdot (D/A) = 0$, that is $\mathfrak{g} \cdot D \subset A$.
Thus $X \cdot \phi = ad_Y$ for some $Y \in SR$.
For $Y \in SR$, $Y \cdot \phi = -c\, ad_Y = -ad_Y$. Thus $X \cdot \phi = (-Y) \cdot \phi$.
Thus $\phi$ induces projecting $X$ to $-Y \in SR$.

**Consider $\mathfrak{g}_0 \coloneqq \{X \in \mathfrak{g} : X \cdot \phi = 0\}$.**
$\mathfrak{g}_0$ is a subalgebra: $[X, Y] \cdot \phi = X \cdot Y \cdot \phi - Y \cdot X \cdot \phi = 0$.
Using the above notation, $X = (X + Y) - Y$ where $X + Y \in \mathfrak{g}_0$ and $-Y \in SR$.
Hence $\mathfrak{g} = SR + \mathfrak{g}_0$.

Also need $SR \cap \mathfrak{g}_0 = 0$.
If $X \in SR \cap \mathfrak{g}_0$, then $X \cdot \phi = 0$ ($X \in \mathfrak{g}_0$) and $X \cdot \phi = -ad_X$ ($X \in SR$).
Thus $ad_X = 0$. ($\mathbb{C} \cdot X$) is an ideal contained in $SR$, and hence either =0 or $SR$.
$(\mathbb{C} \cdot X) \neq SR$ since $[SR, \mathfrak{g}] \neq 0$ in this case.