Representations

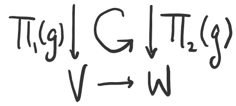
Thursday, February 15, 2018 8:48 AM

**Why representation?**
Represent a group by matrices. Linear actions are the simplest symmetry.
Physics: force (principle bundle) and matter (section of vector bundle).

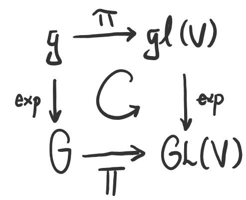
**Complex representation**: $\Pi: G \to GL(V)$ where $V$ is a complex vector space.
For Lie algebras, $\pi: \mathfrak{g} \to \mathfrak{gl}(V)$.
Recall group homomorphism induces algebra homomorphism (and vice versa if $G$ is simply connected).
Thus $\boldsymbol{\Pi}$ **gives** $\boldsymbol{\pi} = \left.\frac{\boldsymbol{d}}{\boldsymbol{dt}}\right|_{\boldsymbol{t=0}} \boldsymbol{\Pi}(\boldsymbol{e^{tx}})$. Also $\pi \circ \mathrm{Ad}_g = \mathrm{Ad}_{\Pi(g)} \circ \pi = \Pi(g) \cdot \pi \cdot \Pi(g)^{-1}$.

**Examples**: standard representations of matrix groups; adjoint representations.

**Faithful**: the homomorphism is injective.
**Irreducible**: V has no non-trivial invariant subspace.
**Unitary**: there is a Hermitian metric on $V$ such that $\Pi: G \to U(V)$. (Then can take $\perp$ of invariant subspace)
**Morphism** between representation: $V \to W$ which commutes with the action (intertwining).



**Suppose G is connected**.



**1. $\boldsymbol{\Pi}$ is irreducible if and only if $\boldsymbol{\pi}$ is**:
$W \subset V$ invariant under $G$ => invariant under $\mathfrak{g}$.
If $W$ is invariant under $\mathfrak{g}$, then it is invariant under
$\exp tX \in G$. But any element in $G$ can be written as product of these.

**2. $\boldsymbol{\Pi_1} \cong \boldsymbol{\Pi_2}$ if and only if $\boldsymbol{\pi_1} \cong \boldsymbol{\pi_2}$**:
=> is obvious since $V \to W$ intertwines with the group actions implies it intertwines with the algebra actions.
<=) $V \to W$ intertwines with $\pi_i(X)$, and hence $\exp \pi_i(X) = \Pi(\exp X)$, and hence $\Pi(g)$ for arbitrary g.

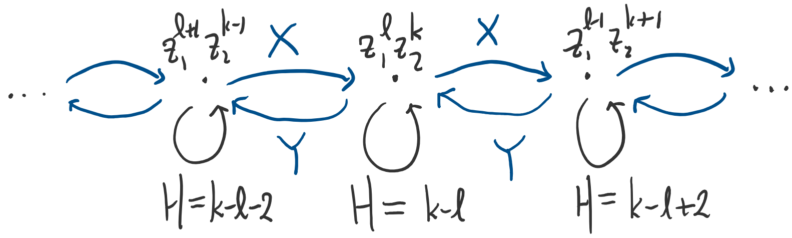
**3. Given $\boldsymbol{\Pi}$. $\boldsymbol{\Pi}: \boldsymbol{G} \to \boldsymbol{U}(\boldsymbol{V})$ <-> $\boldsymbol{\pi}: \boldsymbol{\mathfrak{g}} \to \boldsymbol{\mathfrak{u}}(\boldsymbol{V})$**:
-> we already have $\pi: \mathfrak{g} \to \mathfrak{gl}(V)$. It is obvious that the image is contained in $\mathfrak{u}(V)$.
<- $\Pi(\exp X) = \exp(\pi(X)) \in U(V)$. Any $g \in G$ is a product of $\exp X$.

**Most important example**:
$SU(2) \to GL(\mathbb{C}^2)$, and hence $\boldsymbol{SU(2)} \to \boldsymbol{SU}\left(\boldsymbol{Sym^m}\left(\left(\mathbb{C}^2\right)^*\right)\right)$. Explicitly
$g \cdot f = (g^{-1})^* f = f(g^{-1} \cdot z)$.
Induces $\mathfrak{su}(2) \to \mathfrak{su}\left(Sym^m\left(\left(\mathbb{C}^2\right)^*\right)\right)$.

$$X \cdot f = \left.\frac{d}{dt}\right|_{t=0} f\left(g_t^{-1} \cdot z\right) = \left.df\right|_z \cdot (-X \cdot z).$$

Complexify: $\mathfrak{su} \otimes \mathbb{C} = \mathfrak{sl}_{\mathbb{C}}$ (any complex matrix is a sum of Hermitian and skew-Hermitian matrices.)
Have $\boldsymbol{\mathfrak{sl}(2, \mathbb{C})} \to \boldsymbol{\mathfrak{sl}}\left(\boldsymbol{Sym^m}\left(\left(\mathbb{C}^2\right)^*\right)\right)$ defined by the same formula.
Suppose $f = z_1^l z_2^k$ $(l + k = m)$. $df = l z_1^{l-1} z_2^k \, dz_1 + k z_1^l z_2^{k-1} dz_2$.
Let $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$X \cdot z_1^l z_2^k = l z_1^{l-1} z_2^k \, (-z_2) = -l z_1^{l-1} z_2^{k+1}.$
$Y \cdot z_1^l z_2^k = k z_1^l z_2^{k-1} \, (-z_1) = -k z_1^{l+1} z_2^{k-1}.$
$H \cdot z_1^l z_2^k = l z_1^{l-1} z_2^k \, (-z_1) + k z_1^l z_2^{k-1} (z_2) = (-l+k) z_1^l z_2^k.$
Eigenspace decomposition of H:
$Sym^m\left(\left(\mathbb{C}^2\right)^*\right) = \bigoplus_{l+k=m} \mathbb{C} \cdot z_1^l z_2^k$ with eigenvalues $k-l$.




**$Sym^m\left(\left(\mathbb{C}^2\right)^*\right)$ is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$:**
Suppose W is an invariant subspace and $0 \neq f \in W$. Apply X enough times, it becomes zero.
$0 \neq X^q \cdot f \in \mathbb{C} \cdot z_2^m$ for some $q$. Hence W contains $\mathbb{C} \cdot z_2^m$. Now take $Y^p$, then W contains everything.

**Classification of irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$:**
it must be isomorphic to $V_{m+1} = Sym^m\left(\left(\mathbb{C}^2\right)^*\right)$.
**Proof:**
Have the basis $\{H, X, Y\}$.
For $H$, there is an eigenvalue $a$ and a non-zero eigenvector $v$.
$X \cdot v$ is an eigenvector of $H$ with eigenvalue $a+2$:
$H \cdot (X \cdot v) = X \cdot H \cdot v + 2\, X \cdot v = (a+2)\,(X \cdot v).$
Similarly $Y \cdot v$ is an eigenvector of $H$ with eigenvalue $a-2$:

Since finite-dimensional, $v_0 := X^N \cdot v \neq 0$ but $X^{N+1} \cdot v = 0$. (Such $v_0$ is called to be a highest-weight vector. Its H-eigenvalue $\lambda_0$ is called to be the highest weight.)
Take $v_k = Y^k v_0$ which are eigenvectors of H with distinct eigenvalues $\lambda_0 - 2k$.
$X \cdot v_0 = 0.$
$X \cdot v_1 = Y \cdot X \cdot v_0 + H \cdot v_0 = \lambda_0 v_0.$
$X \cdot v_2 = Y \cdot X \cdot v_1 + H \cdot v_1 = (2\lambda_0 - 2) v_1.$
$X \cdot v_k = \left(k\lambda_0 - 2(1 + \cdots + k - 1)\right) v_{k-1} = k(\lambda_0 - (k-1)) v_{k-1}.$
For certain k, $v_k \neq 0$ but $v_{k+1} = 0$. $X \cdot v_{k+1} = 0$ and so $k = \lambda_0$.
(Up to this point, have not used irreducible.)

$\mathrm{Span}\{v_0, \ldots, v_{\lambda_0}\}$ is invariant. Hence it must be the whole. It is isomorphic to $Sym^{\lambda_0}\left(\left(\mathbb{C}^2\right)^*\right)$.

Can conclude the following for **not necessarily irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$:**
1. All eigenvalues of H are integers. If an H-eigenvector v has $X \cdot v = 0$, then the eigenvalue is non-negative.
2. $\pi(X), \pi(Y)$ are nilpotent.
3. If $k$ is an H-eigenvalue, so are $-|k|, \ldots, |k|$.

Since $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, these are all the irreducible representations of $\mathfrak{so}(3)$. These are all the irreducible representations of SU(2) which is simply connected. But SO(3) is not!

**Prop**: $V_{m+1}$ integrates to a **representation of SO(3)** if and only if $m$ is even.
**Proof:** ($m = 0$ is the trivial representation.)
Recall $ad: \mathfrak{su}(2) \cong \mathfrak{so}(3)$ which sends
$E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, E_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, E_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to
$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, F_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
$V_{m+1} = \mathrm{Span}\{v_0, \dots, v_m\}$, $v_j$ are H-eigenvectors with eigenvalues $m - 2j$.
$E_1 = \frac{\boldsymbol{i}}{2} H$ has eigenvalue (m-2j)$\mathbf{i}/2$.
If m is odd, $e^{2\pi E_1} = e^{2\pi F_1}$ has eigenvalue -1. But $e^{2\pi F_1} = \mathrm{Id}$! Contradiction!

Suppose m is even. Want to say $\mathrm{SU}(2) \to \mathrm{GL}(\mathrm{V}_{\mathrm{m+1}})$ has kernel containing $\pm\mathrm{Id}$, and hence descends to $\mathrm{SO}(3) \to \mathrm{GL}(\mathrm{V}_{\mathrm{m+1}})$. $-\mathrm{Id}_{\mathrm{SU}(2)} = e^{2\pi E_1}$ has eigenvalue 1 on $v_j$ for all j, and hence acts as $\mathrm{Id}_{\mathrm{GL}(\mathrm{V}_{\mathrm{m+1}})}$.

**Exercises. (Section 4.9)**

2. Show that the adjoint representation and the standard representation of $\mathfrak{so}(3)$ are isomorphic.
13. Let $\pi$ be a representation of $\mathfrak{sl}(2, \mathbb{C})$. Show that the eigenvalues of $\pi(H)$ for $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are integers (by using $e^{2\pi i H} = Id \in SU(2)$).