Why representation?
Represent a group by matrices. Linear actions are the simplest symmetry.
Physics: force (principle bundle) and matter (section of vector bundle).

**Complex representation:** $\Pi: G \to GL(V)$ where $V$ is a complex vector space.
For Lie algebras, $\pi: g \to gl(V)$.
Recall group homomorphism induces algebra homomorphism (and vice versa if $G$ is simply connected).
Thus $\Pi$ gives $\pi = \frac{d}{dt} \bigg|_{t=0} \Pi(e^{tx})$. Also $\pi \circ \text{Ad}_g = \text{Ad}_{\Pi(g)} \circ \pi = \Pi(g) \cdot \pi \cdot \Pi(g)^{-1}$.

**Examples:** standard representations of matrix groups; adjoint representations.

**Faithful:** the homomorphism is injective.
**Irreducible:** $V$ has no non-trivial invariant subspace.
**Unitary:** there is a Hermitian metric on $V$ such that $\Pi: G \to U(V)$. (Then can take $\perp$ of invariant subspace)
**Morphism** between representation: $\mathcal{V} \to \mathcal{W}$ which commutes with the action (intertwining).

Suppose $G$ is connected.

1. $\Pi$ is irreducible if and only if $\pi$ is:
   $W \subset V$ invariant under $G \implies$ invariant under $g$.
   If $W$ is invariant under $g$, then it is invariant under $\exp tX \in G$. But any element in $G$ can be written as product of these.

2. $\Pi_1 \cong \Pi_2$ if and only if $\pi_1 \cong \pi_2$:
   $\implies$ is obvious since $\Pi \to W$ intertwines with the group actions implies it intertwines with the algebra actions.
   $\implies$ $V \to W$ intertwines with $\pi_i(X)$, and hence $\exp \pi_i(X) = \Pi(\exp X)$, and hence $\Pi(g)$ for arbitrary $g$.

3. Given $\Pi$. $\Pi: G \to U(V)$ $\iff$ $\pi: g \to u(V)$:
   $\implies$ we already have $\pi: g \to gl(V)$. It is obvious that the image is contained in $u(V)$.
   $\iff$ $\exp \pi(g) \in U(V)$. Any $g \in G$ is a product of $\exp X$.

**Most important example:**
$SU(2) \to GL(\mathbb{C}^2)$, and hence $SU(2) \to SU\left(\text{Sym}^m\left((\mathbb{C}^2)^*\right)\right)$. Explicitly
$g \cdot f = (g^{-1})^* f = f(g^{-1} \cdot z)$.
Induces $\mathfrak{su}(2) \to \mathfrak{u}\left(\text{Sym}^m\left((\mathbb{C}^2)^*\right)\right)$.

$X \cdot f = \frac{d}{dt} \bigg|_{t=0} f(g e^{X} \cdot z) = df \bigg|_{z} \cdot (-X \cdot z)$.
Complexify: $\mathfrak{su} \otimes \mathbb{C} = \mathfrak{sl}$ (any complex matrix is a sum of Hermitian and skew-Hermitian matrices.)

Have $\mathfrak{su}(2, \mathbb{C}) \to \mathfrak{sl}\left(\text{Sym}^m\left((\mathbb{C}^2)^*\right)\right)$ defined by the same formula.

Suppose $f = z_1^l z_2^k \ (l + k = m)$. $df = l z_1^{l-1} z_2^k \ dz_1 + k z_1^l z_2^{k-1} \ dz_2$.

Let $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
$X \cdot z^l z^k_2 = lz^{l-1} z^k_2 (-z_2) = -lz^{l-1} z^{k+1}_2$.
$Y \cdot z^l z^k_2 = kz^{l} z^{k-1}_2 (-z_1) = -kz^{l+1} z^{k-1}_2$.
$H \cdot z^l z^k_2 = lz^{l-1} z^k_2 (-z_1) + kz^l z^{k-1}_2 (z_2) = (-l + k)z^l z^k_2$.

Eigenspace decomposition of H:

$Sym^m \left( (\mathbb{C}^2)^* \right) = \bigoplus_{l+k=m} \mathbb{C} \cdot z^l z^k_2$ with eigenvalues $k - l$.

Sym$^m \left( (\mathbb{C}^2)^* \right)$ is an irreducible representation of sl(2, $\mathbb{C}$):

Suppose W is an invariant subspace and $0 \neq f \in W$. Apply X enough times, it becomes zero.
$0 \neq X^q \cdot f \in \mathbb{C} \cdot z^m_2$ for some q. Hence W contains $\mathbb{C} \cdot z^m_2$. Now take $Y^p$, then W contains everything.

Classification of irreducible representations of sl(2, $\mathbb{C}$):
it must be isomorphic to $V_{m+1} = Sym^m \left( (\mathbb{C}^2)^* \right)$.

Proof:
Have the basis $\{H, X, Y\}$.
For H, there is an eigenvalue $a$ and a non-zero eigenvector $v$.
$X \cdot v$ is an eigenvector of H with eigenvalue $a + 2$:
$H \cdot (X \cdot v) = X \cdot H \cdot v + 2X \cdot v = (a + 2)(X \cdot v)$.
Similarly $Y \cdot v$ is an eigenvector of H with eigenvalue $a - 2$:

Since finite-dimensional, $v_0 := X^N \cdot v \neq 0$ but $X^{N+1} \cdot v = 0$. (Such $v_0$ is called to be a highest-weight vector. Its H-eigenvalue $\lambda_0$ is called to be the highest weight.)
Take $v_k := Y^k v_0$ which are eigenvectors of H with distinct eigenvalues $\lambda_0 - 2k$.
$X \cdot v_0 = 0$.
$X \cdot v_1 = Y \cdot X \cdot v_0 + H \cdot v_0 = \lambda_0 v_0$.
$X \cdot v_2 = Y \cdot X \cdot v_1 + H \cdot v_1 = (2\lambda_0 - 2)v_1$.
$X \cdot v_k = (k\lambda_0 - 2(1 + \cdots + k - 1))v_{k-1} = k(\lambda_0 - (k - 1))v_{k-1}$.
For certain $k$, $v_k \neq 0$ but $v_{k+1} = 0$. $X \cdot v_{k+1} = 0$ and so $k = \lambda_0$.
(Up to this point, have not used irreducible.)

$\text{Span}\{v_0, \ldots, v_{\lambda_0}\}$ is invariant. Hence it must be the whole. It is isomorphic to $Sym^{\lambda_0} \left( (\mathbb{C}^2)^* \right)$.

Can conclude the following for not necessarily irreducible representations of sl(2, $\mathbb{C}$):
1. All eigenvalues of H are integers. If an H-eigenvector v has $X \cdot v = 0$, then the eigenvalue is non-negative.
2. $\pi(X), \pi(Y)$ are nilpotent.
3. If $k$ is an H-eigenvalue, so are $-|k|, \ldots, |k|$.

Since $su(2) \cong so(3)$, these are all the irreducible representations of $so(3)$.
These are all the irreducible representations of SU(2) which is simply connected. But SO(3) is not!
Prop: $V_{m+1}$ integrates to a representation of $SO(3)$ if and only if $m$ is even.

Proof: $(m = 0$ is the trivial representation.)
Recall $ad: su(2) \cong so(3)$ which sends
\[
E_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
E_2 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
E_3 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
\[
F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},
F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},
F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

$E_1 = {\frac{i}{2}} H$ has eigenvalue $(m-2j)i/2.$
If $m$ is odd, $e^{2\pi E_1} = e^{2\pi F_1}$ has eigenvalue -1. But $e^{2\pi F_1} = \text{Id}$! Contradiction!

Suppose $m$ is even. Want to say $SU(2) \to GL(V_{m+1})$ has kernel containing $\pm \text{Id}$, and hence descends to $SO(3) \to GL(V_{m+1})$. $-\text{Id}_{SU(2)} = e^{2\pi E_1}$ has eigenvalue 1 on $v_j$ for all $j$, and hence acts as $\text{Id}_{GL(V_{m+1})}$.

Exercises. (Section 4.9)
2. Show that the adjoint representation and the standard representation of $so(3)$ are isomorphic.
13. Let $\pi$ be a representation of $sl(2, \mathbb{C})$. Show that the eigenvalues of $\pi(H)$ for $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are integers (by using $e^{2\pi i H} = \text{Id} \in SU(2)$).