Schur's Lemma

Monday, February 26, 2018 8:34 AM

Completely reducible representation: direct sum of irreducible ones.
Completely reducible Lie group/algebra: every representation is completely reducible.

Non-example: $\mathbb{R} \to \mathrm{GL}(2, \mathbb{C}), x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$
Preserve e_1, but there is no other invariant subspace.

If V is completely reducible, then given any invariant $U \subset V$, there is U' such that $U \oplus U' = V$:
$V = V_1 \oplus \cdots \oplus V_k$.
Note: either $V_i \subset U$ or $V_i \cap U = \{0\}$.
There is $V_i \not\subset U$ (otherwise $U = V$ and done), say $i = 1$.
If $U + V_1 = V$, done.
Otherwise, have another $V_i \not\subset U$, say $i = 2$. Keep on doing this, until $U + V_1 + \cdots + V_{j-1} = V$. Take $U' = V_1 \oplus \cdots \oplus V_{j-1}$. Done.

Every invariant subspace U of a completely reducible V is completely reducible:
$U \cong V/U' \cong V_j \oplus \cdots \oplus V_k$ in the above notation.

If Π or π is unitary, then it is completely reducible:
Take invariant subspaces and its orthogonal complement.

If G is compact, then it is completely reducible:
Any representation is unitary with respect to a metric. Take arbitrary metric h, and take the average to make it G-invariant:

$$\langle v, w \rangle_G \coloneqq \int_G \langle \Pi(\mathrm{g}) \cdot v, \Pi(\mathrm{g}) \cdot w \rangle \ \nu_G$$

where ν_G is a non-zero right-G-invariant top form on G. It is G-invariant:

$$\langle \Pi(h) \cdot v, \Pi(h) \cdot w \rangle_G \coloneqq \int_G \langle \Pi(\mathrm{g}) \cdot \Pi(h) \cdot v, \Pi(\mathrm{g}) \cdot \Pi(h) \cdot w \rangle \ \nu_G$$

$$= \int_G \langle \Pi(\mathrm{R_h} \cdot \mathrm{g}) \cdot v, \Pi(\mathrm{R_h} \cdot \mathrm{g}) \cdot w \rangle \ \nu_G$$

$$= \int_G \langle \Pi(\mathrm{g}) \cdot v, \Pi(\mathrm{g}) \cdot w \rangle \ (R_h^*)^{-1} \cdot \nu_G = \int_G \langle \Pi(\mathrm{g}) \cdot v, \Pi(\mathrm{g}) \cdot w \rangle \ \nu_G = \langle v, w \rangle_G.$$

Hence $\mathfrak{sl}(\mathrm{n}, \mathbb{C}) = \mathfrak{su}(n) \otimes \mathbb{C}$ is completely reducible.

Schur's Lemma:

If V, W are irreducible representations, then any non-zero morphism $\phi: V \to W$ is an isomorphism. Any ϕ_1 and $\phi_2 \neq 0$ are related by $\phi_1 = \lambda \phi_2$.
If V=W, then $\phi = \lambda \cdot Id$.
Proof:
Consider Ker ϕ, which is invariant since ϕ intertwines with G-action. V is irreducible and $\phi \neq 0$ imply Ker $\phi = 0$.
Im ϕ is also invariant. W is irreducible and and $\phi \neq 0$ imply Im $\phi = W$.

If V=W, consider eigenvalues of ϕ. There exists an eigenvalue λ since we work over $\mathbb{C}$. The eigenspace is G-invariant since ϕ is intertwining. V is irreducible implies the eigenspace is V.

For $\phi_1, \phi_2: V \to W$, consider $\phi_2^{-1} \circ \phi_1: V \to V$, which is $\lambda \cdot Id$.

Corollary: for a commutative Lie group/algebra, irreducible representation is one-dimensional.
Proof: $\Pi(g): V \to V$ for any g is a morphism: $\Pi(h) \circ \Pi(g) = \Pi(g) \circ \Pi(h)$.
Schur's lemma gives $\Pi(g) = \lambda_g \cdot \mathrm{Id}$. Any subspace is invariant. Irreducible => 1d.

Same proof gives
Corollary: $\Pi(g) = \lambda_g \cdot \mathrm{Id}$ if g belongs to center and Π is irreducible.

Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & \theta \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, \theta \in \mathbb{R} \right\}$$

which is universal cover of
$G = \mathbb{R} \times \mathbb{R} \times \mathbb{S}^1$ with
$(x_1, y_1, u_1) \cdot (x_2, y_2, u_2) = (x_1 + x_2, y_1 + y_2, e^{2\pi i\, x_1 y_2}\, u_1\, u_2)$.
$u = e^{2\pi i \theta}$. (Inverse of (x, y, u) is $(-x, -y, e^{2\pi i\, xy} u^{-1})$.)

$$\boldsymbol{G = H/N} \text{ where } N = \left\{ \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\} \text{ (Kernel).}$$

$(x_1, y_1, u_1) \cdot (x_2, y_2, u_2) \cdot (x_1, -y_1, e^{2\pi i x_1 y_1} u_1^{-1})$
$= (x_2, y_2, e^{-2\pi i (x_1 + x_2) y_1} e^{2\pi i\, x_1\, y_2} e^{2\pi i x_1 y_1} u_2)$
$= (x_2, y_2, e^{2\pi i (x_1 y_2 - x_2 y_1)} u_2)$.
Thus **center** $Z(H)$ **is** $\boldsymbol{x = y = 0}$.

Theorem: G has no faithful representation.

Proof: Lift to representation $\widetilde{\Pi}$ of H with $\mathrm{Ker} \supset N$.

Claim: $\mathrm{Ker}(\widetilde{\Pi}) \supset Z(H)$, and hence $\mathrm{Ker}(\Pi) \supset Z(G) = \{0\} \times \{0\} \times \mathbb{S}^1$, not faithful.

Proof of claim: consider the Lie algebras. Want to see $\mathrm{Ker}(\pi) \ni \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = X_3$. Then $\mathrm{Ker}(\widetilde{\Pi}) \ni e^{tX_3}$.

$e^{X_3} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in N$. Thus $e^{\pi(X_3)} = \widetilde{\Pi}(e^{X_3}) = \mathrm{Id}$. Hence $e^{k\,\pi(X_3)} = \mathrm{Id}$ for all $k \in \mathbb{Z}$.

But $e^{k\,\pi(X_3)}$ is polynomial in k since $\pi(X_3)$ is nilpotent. Then
the equation in k has infinitely many roots implies the polynomial is identically zero.
$\pi(X_3)$ is nilpotent: need to show it only has zero eigenvalue. Note that $X_3 = [X_1, X_2]$.
Take an eigenspace U of X_3. It is invariant since X_1, X_2 commutes with X_3.
$\mathrm{tr}(\pi(X_3)|_U) = 0$. Thus the eigenvalue must be zero.

Exercises. (Section 4.9)

5. Consider the standard representation of the Heisenberg group acting on $\mathbb{C}^3$. Determine all the invariant subspaces. Is the standard representation completely reducible?
11. Let V be an irreducible representation over $\mathbb{C}$. Show that every non-trivial invariant subspace of $V \oplus V$ is of the form $\{(\lambda_1 v, \lambda_2 v) : v \in V\} \cong V$ for some $(\lambda_1, \lambda_2) \in \mathbb{C}^2 - \{0\}$.