Basis:

\[
H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

\[
X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then \([H_1, H_2] = 0, [H_1, X_1] = 2X_1, [H_1, X_2] = -X_2, [H_1, X_3] = X_3, [H_2, X_1] = -X_1, [H_2, X_3] = X_3, [H_2, Y_1] = -2Y_1,\) and so on;

\([X_1, Y_1] = H_1, [X_2, Y_2] = H_2, [X_3, Y_3] = H_1 + H_2, [X_1, X_2] = X_3, [Y_1, Y_2] = -Y_3\) and so on.

Given a representation \(\pi\), consider simultaneous eigenspaces of \(\pi(H_i)\): first take an eigenspace \(U\) of \(\pi(H_1)\); since \([\pi(H_1), \pi(H_2)] = 0\), \(\pi(H_2)\) preserves \(U\) and has an eigenspace in \(U\).

Given a simultaneous eigenvector \(v\), have eigenvalues \(m_i\) of \(\pi(H_i)\).

\(\mu = (m_1, m_2) \in \mathfrak{h}^*\) is called a \textbf{weight} for \(\pi\), where \(\mathfrak{h} = \text{Span}(H_1, H_2)\).

The space of all such simultaneous eigenvectors is called the \(\mu\)-\textbf{weight space}. Its dimension is called to be the multiplicity of \(\mu\).

\(m_i\) \textbf{are integers}: restrict the representation to \(\langle H_i, X_i, Y_i \rangle \cong \mathfrak{sl}(2, \mathbb{C})\).

Now apply the above concept to the adjoint representation. The corresponding non-zero weights are called \textbf{roots}; elements in a weight space are called \textbf{root vectors}.

For \(\mathfrak{sl}(2, \mathbb{C})\) the root vectors and roots are

<table>
<thead>
<tr>
<th>(X)</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Y)</td>
<td>-2</td>
</tr>
</tbody>
</table>

For \(\mathfrak{sl}(3, \mathbb{C})\) the root vectors and roots are

| \(X_1\) | \(\alpha_1 = (2, -1)\) |
| \(X_2\) | \(\alpha_2 = (-1, 2)\) |
| \(X_3\) | \((1, 1)\) |
| \(v\) | \((-2, 1)\) |
\begin{tabular}{ |l|l| } 
\hline
$X_3$ & (1,1) \\
$Y_1$ & (-2,1) \\
$Y_2$ & (1,-2) \\
$Y_3$ & (-1,-1) \\
\hline
\end{tabular}

\[\alpha_i\] are called positive simple roots. All other roots are linear combinations of them with coefficients either all non-negative or all non-positive.

**Prop**: Let $Z_\alpha$ be a root vector. For a representation of $\mathfrak{sl}(3, \mathbb{C})$, $\pi(Z_\alpha)$ sends $\mu$-weight space to $(\mu + \alpha)$-weight space.

**Proof**: let $0 \neq \nu \in \mu = (m_1, m_2)$-weight space of $\pi$. \( \alpha = (a_1, a_2) \).

\[
\pi(H_i) \cdot \pi(Z_\alpha) \nu = \pi(Z_\alpha) \cdot \pi(H_i) \nu + \pi([H_i, Z_\alpha]) \nu = m_i \pi(Z_\alpha) \nu + a_i \pi(Z_\alpha) \nu = (m_i + a_i) \pi(Z_\alpha) \nu.
\]

QED

**Def**: The weights $\mu_1$ is **higher** than $\mu_2$ if $\mu_1 - \mu_2 = a \alpha_1 + b \alpha_2$ for some $a, b \in \mathbb{R}_+$. It gives a partial ordering. ($\alpha_1$ is neither higher nor lower than $\alpha_2.$)

**Highest weight representation with weight $\mu$**: There exists a weight vector $\nu \neq 0$ corresponding to $\mu$ such that $\pi^{(j)}(\nu) = 0$ for all $j$, and $\nu$ is cyclic (that is $V = g \cdot \nu$).

**By definition $\mu$ is really the highest weight and it has multiplicity one**: By keep on taking $Y_i$ on $\nu$, get an invariant subspace which must be $V$.

**it is invariant**: A product of operations can always be expressed in terms of $\pi(Y_1)^{p_1} \pi(Y_2)^{p_2} \pi(Y_3)^{p_3} \pi(H_1)^{q_1} \pi(H_2)^{q_2} \pi(H_3)^{q_3} \pi(X_1)^{r_1} \pi(X_2)^{r_2} \pi(X_3)^{r_3}$. Acting on $\nu$, it becomes scaling of $\pi(Y_1)^{p_1} \pi(Y_2)^{p_2} \pi(Y_3)^{p_3}.$

$Y_i$ decrease the weight. Hence $\mu$ is the unique highest weight, and the $\mu$-weight space is one-dimensional: $\mathbb{C} \cdot \nu$.

**CAUTION: $V$ is cyclic does not imply it is irreducible**: For instance take $V_2 \oplus V_3$ of $\mathfrak{sl}(2, \mathbb{C})$. Then $\nu_2 + \nu_3$ is cyclic (where $\nu_i$ are highest weight vector of $V_i$).

Indeed irreducible $\iff$ every non-zero vector is cyclic.

**Irreducible $\iff$ highest weight representation.**

**Proof.**
Irreducible $V$ is a direct sum of weight spaces:
There exists a $\mu$-weight space $V_\mu$ over $\mathbb{C}$. $Z_\alpha$ sends $V_\mu$ to $V_{\mu+\alpha}$ (and $H_i$ preserve $V_\mu$). Then keep on taking $Z_\alpha$, get an invariant subspace which is $V$ itself. $V_{\mu_1} \cap V_{\mu_2} = \{0\}$ if $\mu_1 \neq \mu_2$.

Since $V$ is finite-dimensional, there must be a highest weight. A corresponding weight vector $v$ must have $\mathfrak{h}(v) v = 0$. $v$ is cyclic since $V$ is irreducible. Hence $V$ is a highest weight representation.

\[ \leq \]
Any finite dimensional representation of $\mathfrak{sl}(3,\mathbb{C})$ corresponds to that of $SU(3)$ which is simply connected and compact. Hence it must be completely reducible. Each irreducible part is a direct sum of weight spaces. Hence the highest weight space, which has dimension one, must belong to one irreducible part. But it is cyclic, and hence the whole $V$ is that part.

QED.

**Theorem:**
Irreducible representation $V$ of $\mathfrak{sl}(3,\mathbb{C}) \leftrightarrow (m_1, m_2) \in \mathbb{Z}_{\geq 0}^2$ where the correspondence is given by taking the highest weight.

**Proof:**

$\rightarrow$

Take the highest weight.
$m_i$ are non-negative:
Restrict to $\langle H_i, X_i, Y_i \rangle \cong \mathfrak{sl}(2,\mathbb{C})$.

This is injective:
Suppose $V$ and $W$ have the same highest weight with weight vectors $v$ and $w$. Consider the subspace $U$ generated by $(v, w) \in V \oplus W$. $(v, w)$ is a weight vector (since $v$ and $w$ have the same weight) which is highest cyclic. Hence $U$ is irreducible. The projection maps $U \to V$ and $U \to W$ are morphisms and non-zero, and hence are isomorphisms by Schur's Lemma.

This is surjective:
Standard representation $V = \mathbb{C}^3$: since
\[ H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

the standard basic vectors are weight vectors with \( \mu = (1,0), (-1,1), (0,-1) \). Recall \( \alpha_1 = (2, -1), \alpha_2 = (-1,2) \). Hence (1,0) is the highest weight and \( e_1 \) is a highest weight vector which is cyclic.

**Dual of standard representation** \( V^* \). The action is right multiplication by \(-X\) on row vectors. The standard row vectors have weights \( \mu = (-1,0), (1, -1), (0,1) \). (0,1) is the highest weight and \( e_3^* \) is a highest weight vector.

Then consider \( V \otimes^{m_1} \otimes (V^*) \otimes^{m_2} \).

\( \nu_{m_1,m_2} = e_1^{\otimes m_1} \otimes (e_3^*)^{\otimes m_2} \) has weight \( (m_1, m_2) \).

Take the invariant subspace generated by \( \nu_{m_1,m_2} \). Then it is a highest weight representation with highest weight \( (m_1, m_2) \). Hence it is irreducible.

QED

**Exercises. (Section 6.9)**

6. Find the weights and multiplicities of the (2,0)-highest weight representation of \( sl(3, \mathbb{C}) \).

8. Show that the space of homogeneous polynomials of degree \( m \) in three variables is the (0,m)-highest weight representation of \( sl(3, \mathbb{C}) \).