\[ \mathfrak{h}_2 := \mathbb{Z} \cdot \{ H_1, H_2 \} \subset i \cdot \mathfrak{su}(3) = \mathfrak{su}(3) \otimes \mathbb{C}. \]

Killing form \( tr(X_1 X_2) \) restricts to be metric on \( i \cdot \mathfrak{su}(3) \).

Use the metric to identify Cartan subalgebra \( \mathfrak{h}_\mathbb{R} \cong \mathfrak{h}^*_\mathbb{R} \).

Standardize the metric (rather than the lattice) of \( \mathfrak{h}_\mathbb{R} \).

\[ \alpha_2 = (-1, 2) \]

\[ \alpha_1 = (2, -1) \]

Direct computation gives: \( H_1 \cdot H_1 = 2, H_1 \cdot H_2 = -1 \).

\[ H_1 \leftrightarrow 2H_1^* - H_2^* = \alpha_1, \quad H_2 \leftrightarrow 2H_2^* - H_1^* = \alpha_2. \]

\[ H_1^* \leftrightarrow \frac{2H_1 + H_2}{3}, \quad H_2^* \leftrightarrow \frac{2H_2 + H_1}{3}. \]

\[ \angle(H_1^*, H_2^*) = \frac{\pi}{3}. \]

Moreover \( \alpha_i \cdot H_j^* = \delta_{ij} \).

SU(3) has adjoint action on \( \mathfrak{su}(3) \) and hence \( \mathfrak{sl}(3, \mathbb{C}) \).

N: subgroup of SU(3) preserving \( \mathfrak{h} \).

Z: subgroup of SU(3) that fixes every element of \( \mathfrak{h} \).

Z is a normal subgroup of N.

\( W := N/Z \): the Weyl group for SU(3).

**Theorem:** For any representation, the set of weights has W-symmetry. Namely,

if \( \lambda \in \mathfrak{h}^* \) is a weight, then \( w \cdot \lambda \) is also a weight with the same multiplicity.

**Proof:**

For \( \nu \in V_\lambda \): \( \lambda \)-weight space,

\[ H \cdot \nu = (\lambda, H) \nu \quad \forall H \in \mathfrak{h}. \]

\( w = [g] \) for \( g \in N \subset SU(3) \). \( w \) acts on \( \mathfrak{h}^* \) by

\[ (w \cdot \lambda, H) = \lambda(Ad_g^{-1} \cdot H) \]

\( (\lambda, g^{-1} H g) \).
Replace $H$ by $g^{-1}Hg \in \mathfrak{h}$,
$$g^{-1}Hg \cdot v = (\lambda, g^{-1}Hg) v = (w \cdot \lambda, H) v.$$  
Thus $H \cdot (g \cdot v) = (w \cdot \lambda, H)(g \cdot v)$. (Here we have integrated the action of $\mathfrak{su}(3)$ to $SU(3)$).
Then $g \cdot V_\lambda$ is the $(w \cdot \lambda)$-weight space.

**Prop:** $W$ is the permutation group on three elements (namely the three weights of the standard representation).

**Proof:**
Recall the standard representation $\mathbb{C}^3$ of $SU(3)$ has weight spaces $\mathbb{C} \cdot e_i$ for $i = 1,2,3$. $W$ permutes the weight spaces. This gives $W \rightarrow S_3$.

**Surjective:**
Given any permutation, we can realize it by a permutation matrix $U$ which permutes the basic vectors. (It lies in $SU(3)$ since it preserves metric. Take a basic vector $e$ to $-e$ if necessary to keep orientation).
$[U] \in W$: if $H$ is diagonal, $UHU^{-1}$ is still diagonal since $U^{-1}e_i$ is a basic vector and hence $H$ acts by scaling. Thus $U$ preserves $\mathfrak{h}$.

**Injective:**
If $[U] \mapsto 1$, then $U$ is diagonal. Then $UHU^{-1} = H$ for any $H \in \mathfrak{h}$, and hence $[U] = 1 \in W$.
QED

**Conclusion:** $W \cong S_3$ is the symmetry group of the equilateral triangle, which acts by reflections about $\alpha_i^\perp$.

**Integral structure:**
$\mathfrak{h}_\mathbb{Z} = \mathbb{Z} \cdot \{H_1, H_2\} \subset \mathfrak{h}_\mathbb{R}$. $\mathfrak{h}_\mathbb{Z}^* = \text{Hom}(\mathfrak{h}_\mathbb{Z}, \mathbb{Z}) \subset \mathfrak{h}_\mathbb{R}^*$.

**Dominant structure:**
$h_{\mathbb{Z},+} = \mathbb{Z}_{\geq 0} \cdot \{H_1, H_2\}$. $h_{\mathbb{Z},+}^* = \text{Hom}(h_{\mathbb{Z},+}, \mathbb{Z}_{\geq 0})$.

**Theorem:**
For the highest weight representation $V_\mu$,
$
\lambda \in h_{\mathbb{Z}}^*$ is a weight (with non-zero multiplicity) if and only if $\lambda \in (\mu + \mathbb{Z} \cdot \langle \alpha_1, \alpha_2 \rangle) \cap \text{Conv}(W \cdot \mu)$.

**Proof:**
$\Rightarrow$
$V_\mu$ is spanned by $Y_{k_1} \ldots Y_{k_j} \cdot \nu$ where $\nu$ is the highest weight vector.
These are weight vectors with weights $\lambda \in \mu - \mathbb{Z} \cdot \langle \alpha_1, \alpha_2 \rangle$.
$w \cdot \lambda \in h_{\mathbb{Z},+}^*$ for some $w \in W$. Also $w \cdot \lambda \leq \mu$ since it is still a weight.
Hence it belongs to the shaded quadrilateral.
$w_{\alpha_i}$ acts as reflection about $\alpha_i^\perp$. Hence the vertices of the quadrilateral are $0, \mu, \frac{w_{\alpha_1} \cdot \mu + \mu}{2}, \frac{w_{\alpha_2} \cdot \mu + \mu}{2}$.
of the quadrilateral are \(0, \mu, \frac{w_{\alpha_1 + \mu}}{2}, \frac{w_{\alpha_2 + \mu}}{2}\) which all belong to \(\text{Conv}(W \cdot \mu)\).

\[\leq\] (no hole in between)

First each element \(w\) in \((\mu - \mathbb{Z}_{\geq 0} \cdot \alpha_i) \cap \text{Conv}(W \cdot \mu)\) (for \(i=1,2\)) is a weight by restricting to \(\langle H_i, X_i, Y_i \rangle \cong \mathfrak{sl}(2, \mathbb{C})\).

Then each element in \((\mathbb{Z}_{\geq 0} \cdot \alpha_j) \cap \text{Conv}(W \cdot \mu)\) (for \(j = 1,2,3\)) is a weight by restricting to \(\langle H_j, X_j, Y_j \rangle \cong \mathfrak{sl}(2, \mathbb{C})\) (where \(H_3 = H_1 + H_2\)). This covers all the elements in \((\mu + \mathbb{Z} \cdot \langle \alpha_1, \alpha_2 \rangle) \cap \text{Conv}(W \cdot \mu)\).

**Exercises. (Section 6.9)**

10. Classify the irreducible representations of \(\mathfrak{sl}(3, \mathbb{C})\) whose sets of weights are invariant under \(-\mathbf{I}\) on \(\mathfrak{h}^*_\mathbb{R}\).

11. For the highest weight representation \(V_{\lambda}\), show that the \((\mu - \alpha_1 - \alpha_2)\)-weight space has multiplicity at most two, and it is spanned by \(Y_1 \cdot Y_2 \cdot v_0\) and \(Y_2 \cdot Y_1 \cdot v_0\) where \(v_0\) is a highest weight vector.