Weyl group of SU(3)

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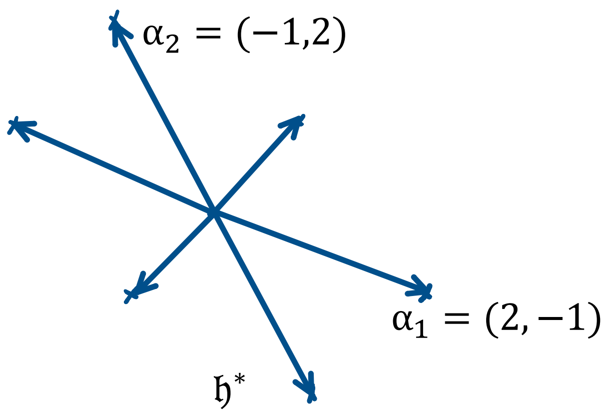
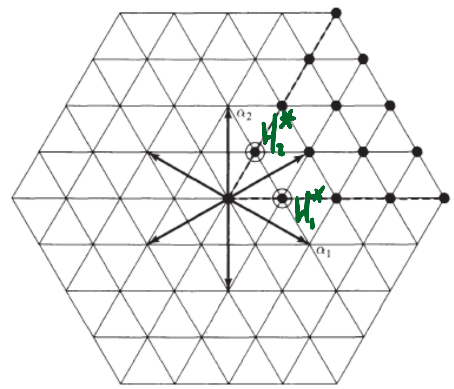
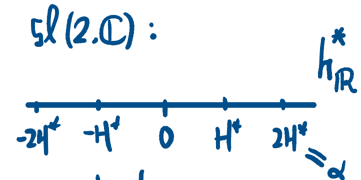
$\mathfrak{h}_{\mathbb{Z}} := \mathbb{Z} \cdot \{\boldsymbol{H_1}, \boldsymbol{H_2}\} \subset i \cdot \mathfrak{su}(3) = \mathfrak{su}(3) \otimes \mathbb{C}$.
Killing form $tr(X_1 X_2)$ restricts to be metric on $i \cdot \mathfrak{su}(3)$.
Use the metric to identify Cartan subalgebra $\mathfrak{h}_{\mathbb{R}} \cong \mathfrak{h}^*_{\mathbb{R}}$.
Standardize the metric (rather than the lattice) of $\mathfrak{h}^*_{\mathbb{R}}$.





$\mathfrak{sl}(2.\mathbb{C})$:



$H \cdot H = 2.$

$H \longleftrightarrow 2H^* = \alpha$

$H^* \cdot H^* = \frac{H}{2} \cdot \frac{H}{2} = \frac{1}{2}.$

$\alpha \cdot H^* = 1.$

Direct computation gives: $H_i \cdot H_i = 2, H_1 \cdot H_2 = -1$.
$H_1 \leftrightarrow 2H_1^* - H_2^* = \alpha_1$, $\quad H_2 \leftrightarrow 2H_2^* - H_1^* = \alpha_2$.
$H_1^* \leftrightarrow \dfrac{2\,H_1 + H_2}{3}, H_2^* \leftrightarrow \dfrac{2\,H_2 + H_1}{3}. \angle(H_1^*, H_2^*) = \dfrac{\pi}{3}.$
Moreover $\alpha_i \cdot H_j^* = \delta_{ij}$.

SU(3) has adjoint action on $\mathfrak{su}(3)$ and hence $\mathfrak{sl}(3, \mathbb{C})$.

N: subgroup of SU(3) preserving $\mathfrak{h}$.
Z: subgroup of SU(3) that fixes every element of $\mathfrak{h}$.
Z is a normal subgroup of N.
$W := N/Z$: the Weyl group for SU(3).

Theorem: For any representation, the set of weights has W-symmetry. Namely,
if $\lambda \in \mathfrak{h}^*$ is a weight, then $w \cdot \lambda$ is also a weight with the same multiplicity.

Proof:
For $v \in V_\lambda$: λ-weight space,
$H \cdot v = (\lambda, H)\, v \;\; \forall H \in \mathfrak{h}$.
$w = [g]$ for $g \in N \subset SU(3)$. w acts on $\mathfrak{h}^*$ by
$(w \cdot \lambda, H) = \lambda(Ad_g^{-1} \cdot H) = (\lambda, g^{-1} H g)$.

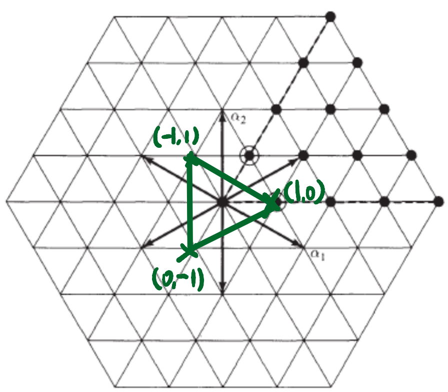
Replace H by $g^{-1}Hg \in \mathfrak{h}$,
$g^{-1}Hg \cdot v = (\lambda, g^{-1}Hg)\, v = (w \cdot \lambda, H)\, v$.
Thus $H \cdot (g \cdot v) = (w \cdot \lambda, H)(g \cdot v)$. (Here we have integrated the action of $\mathfrak{su}(3)$ to $SU(3)$).
Then $g \cdot V_\lambda$ is the $(w \cdot \lambda)$-weight space.

Prop: W is the permutation group on three elements (namely the three weights of the standard representation).

Proof:
Recall the standard representation $\mathbb{C}^3$ of SU(3) has weight spaces $\mathbb{C} \cdot e_i$ for $i = 1,2,3$. W permutes the weight spaces. This gives $W \to S_3$.

Surjective:
Given any permutation, we can realize it by a permutation matrix U which permutes the basic vectors. (It lies in SU(3) since it preserves metric. Take a basic vector e to $-e$ if necessary to keep orientation).
$[U] \in W$: if H is diagonal, UHU^{-1} is still diagonal since $U^{-1}e_i$ is a basic vector and hence H acts by scaling. Thus U preserves $\mathfrak{h}$.

Injective:
If $[U] \mapsto \mathrm{Id}$, then U is diagonal. Then $UHU^{-1} = H$ for any $H \in \mathfrak{h}$, and hence $[U] = 1 \in W$.
QED

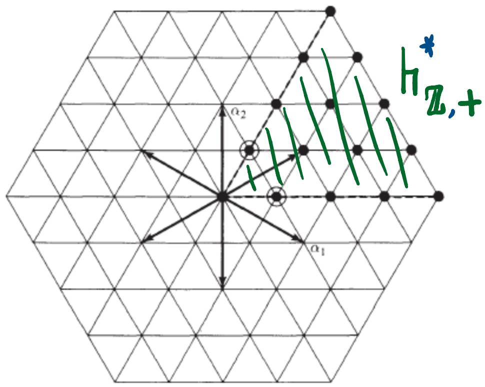


standard representation

Conclusion: $W \cong S_3$ is the symmetry group of the equilateral triangle, which acts by reflections about $\alpha_i^\perp$.

Integral structure:
$\mathfrak{h}_\mathbb{Z} = \mathbb{Z} \cdot \{H_1, H_2\} \subset \mathfrak{h}_\mathbb{R}$. $\mathfrak{h}_\mathbb{Z}^* = \mathrm{Hom}(\mathfrak{h}_\mathbb{Z}, \mathbb{Z}) \subset \mathfrak{h}_\mathbb{R}^*$.
Dominant structure:

$\mathfrak{h}_{\mathbb{Z},+} = \mathbb{Z}_{\geq 0} \cdot \{H_1, H_2\}$. $\mathfrak{h}^*_{\mathbb{Z},+} = \mathrm{Hom}(\mathfrak{h}_{\mathbb{Z},+}, \mathbb{Z}_{\geq 0})$.



Theorem:
For the highest weight representation V_μ,
$\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$ is a weight (with non-zero multiplicity) if and only if $\lambda \in (\mu + \mathbb{Z} \cdot \langle \alpha_1, \alpha_2 \rangle) \cap \mathrm{Conv}(W \cdot \mu)$.

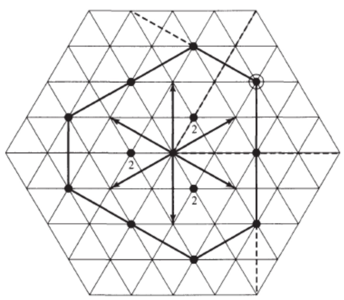


Fig. 5.4. Highest weight (1,2)

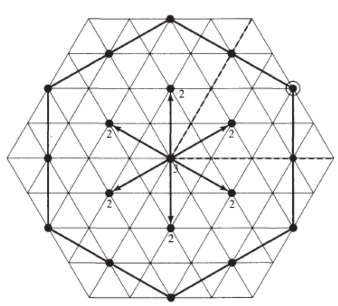
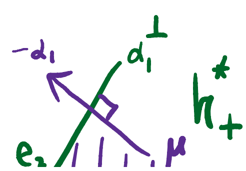
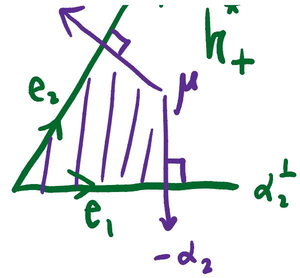


Fig. 5.5. Highest weight (2,2)

Proof:
=>)
V_μ is spanned by $Y_{k_1} \dots Y_{k_j} \cdot v$ where v is the highest weight vector.
These are weight vectors with weights $\lambda \in \mu - \mathbb{Z} \cdot \langle \alpha_1, \alpha_2 \rangle$.
$w \cdot \lambda \in \mathfrak{h}^*_{\mathbb{Z},+}$ for some $w \in W$. Also $w \cdot \lambda \leq \mu$ since it is still a weight.
Hence it belongs to the shaded quadrilateral.
w_{α_i} acts as reflection about $\alpha_i^\perp$. Hence the vertices of the quadrilateral are $0, \mu, \frac{w_{\alpha_1} \cdot \mu + \mu}{2}, \frac{w_2 \cdot \mu + \mu}{2}$



of the quadrilateral are $0, \mu, \frac{w_{\alpha_1}\cdot\mu+\mu}{2}, \frac{w_2\cdot\mu+\mu}{2}$
which all belong to $\mathrm{Conv}(W\cdot\mu)$.



<=) (no hole in between)

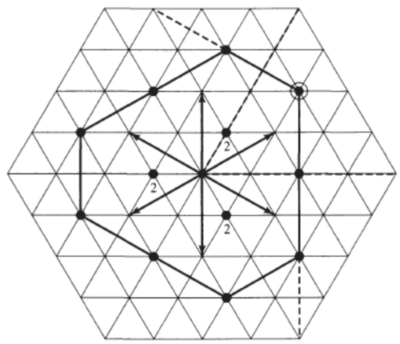


Fig. 5.4. Highest weight (1,2)

First each element w in $(\mu-\mathbb{Z}_{\geq 0}\cdot\alpha_i)\cap\mathrm{Conv}(W\cdot\mu)$ (for i=1,2) is a weight by restricting to $\langle H_i, X_i, Y_i\rangle \cong \mathfrak{sl}(2,\mathbb{C})$.
Then each element in $(w-\mathbb{Z}_{\geq 0}\cdot\alpha_j)\cap\mathrm{Conv}(W\cdot\mu)$ (for $j=1,2,3$) is a weight by restrictly to $\langle H_j, X_j, Y_j\rangle \cong \mathfrak{sl}(2,\mathbb{C})$ (where $H_3 = H_1 + H_2$).
This covers all the elements in $(\mu+\mathbb{Z}\cdot\langle\alpha_1,\alpha_2\rangle)\cap\mathrm{Conv}(W\cdot\mu)$.

Exercises. (Section 6.9)

10. Classify the irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$ whose sets of weights are invariant under $-\mathrm{Id}$ on $\mathfrak{h}^*_{\mathbb{R}}$.
11. For the highest weight representation V_λ, show that the $(\mu-\alpha_1-\alpha_2)$-weight space has multiplicity at most two, and it is spanned by
$Y_1\cdot Y_2\cdot v_0$ and $Y_2\cdot Y_1\cdot v_0$
where v_0 is a highest weight vector.