Abstract root system

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Root system $R \subset E$:

1. R spans E .
2. For $\alpha \in R, \pm\alpha$ are the only multiples of α which belong to R.
3. $s_\alpha \cdot \beta \in R$ for $\alpha, \beta \in R$.
4. $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$.

Weyl group: the group generated by s_α.
Can be identified as a subgroup of the permutation group of R (and hence is finite if R is finite).

If $R \subset E$ and $S \subset F$ are root systems, then so is $R \cup S \subset E \oplus F$.

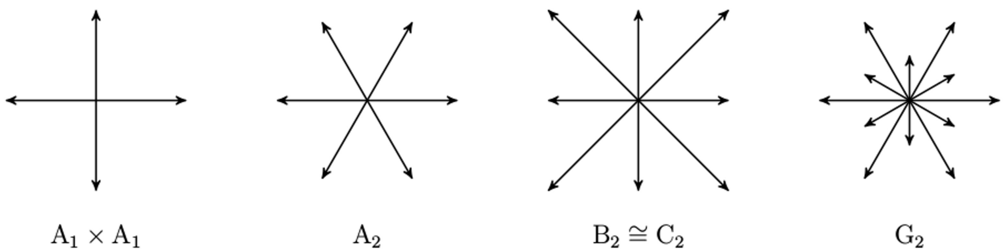
Morphism of root system:
linear map A with $A(R) \subset S$ and commute with Weyl action:
$A(s_\alpha \cdot \beta) = s_{A\alpha} \cdot (A\beta)$.
Note that it may not preserve metric. (Allow scaling. Otherwise too many non-isomorphic root systems.)

Prop. let α, β be linearly independent roots.
WLOG let $|\alpha| \geq |\beta|$. Then either

1. $\langle\alpha,\beta\rangle = 0$.
2. $\langle\alpha,\alpha\rangle = \langle\beta,\beta\rangle$ and the angle between the two lines is $\frac{\pi}{3}$.
3. $\langle\alpha,\alpha\rangle = 2\langle\beta,\beta\rangle$ and the angle between the two lines is $\frac{\pi}{4}$.
4. $\langle\alpha,\alpha\rangle = 3\langle\beta,\beta\rangle$ and the angle between the two lines is $\frac{\pi}{6}$.

Proof:
$m_1 = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ and $m_2 = \frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$.
$m_1 m_2 = 4\cos^2\theta$.
Hence $0 \leq m_1 m_2 \leq 4$.
Five cases: $m_1 m_2 = 0,1,2,3,4$. Remaining is plane geometry.
QED

Cor:
Angle between roots α and β is
strictly obtuse => $\alpha + \beta$ is root.
strictly acute => $\alpha - \beta$ and $\beta - \alpha$ are roots.
Proof:
Consider $s_\alpha \cdot \beta$ which is a root. QED



R$^\vee$: set of all coroots $H_\alpha = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$.
Prop. $R^\vee$ is also a root system and it has the same Weyl group.
$(R^\vee)^\vee = R$.
Proof:
Condition 1 and 2 for root system are obvious.
Direct check that

$$\frac{2H_\alpha}{\langle H_\alpha, H_\alpha\rangle} = \alpha.$$

(Hence $(R^\vee)^\vee = R$.)

$$\frac{2\langle H_\alpha, H_\beta\rangle}{\langle H_\alpha, H_\alpha\rangle} = \langle \alpha, H_\beta\rangle = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}.$$

$$s_{H_\alpha} \cdot H_\beta = s_\alpha \cdot H_\beta = \frac{2}{\langle\beta,\beta\rangle} s_\alpha \cdot \beta = \frac{2}{\langle s_\alpha \cdot \beta, s_\alpha \cdot \beta\rangle} s_\alpha \cdot \beta = H_{s_\alpha \cdot \beta}.$$

QED

Base Δ (set of all positive simple roots):
a subset of R which is a basis of E, and every root is an integer combination in Δ with coefficients of the same sign.

Any two vectors in Δ have right or obtuse angle:

Otherwise $\alpha - \beta$ would be a root, contradicting that all coefficients are the same sign.

Construction of a base:
Take a hyperplane not containing any root of R (take H not in any $\alpha^\perp$, and take $H^\perp$).
Take R^+ to be the roots in one side.

Prop. The set **Δ of indecomposible elements of R^+ is a base**.
(Indecomposible means $\alpha \neq \beta + \gamma$ for any $\beta, \gamma \in R^+$.)
Proof:
Any roots in R^+ is an integer combination of Δ with positive coefficients:
Keep on splitting $\alpha = \beta + \gamma$ for $\beta, \gamma \in R^+$. Must terminate in finite steps: the distance of α from $H^\perp$ is strictly decreasing.

Any other roots are $-\alpha$ for $\alpha \in R^+$. Hence **any root is an integer combination of Δ with coefficients of the same sign**.

Δ is linearly independent:
Suppose $\sum_\alpha c_\alpha \alpha = \sum_\beta d_\beta \beta$ where the coefficients are all positive (and the sums are over disjoint subsets of Δ).
Consider its norm squared: $\sum c_\alpha d_\beta \langle \alpha, \beta \rangle$.

$\langle \alpha, \beta \rangle \leq 0$ for any distict $\alpha, \beta \in \Delta$, and so the above has to be zero:
Otherwise $\alpha - \beta$ and $\beta - \alpha$ would be roots, and one of them belongs to R^+, contradicting that coefficients have the same sign.

Thus $\sum_\alpha c_\alpha \alpha = \sum_\beta d_\beta \beta = 0$ and all coefficients are positive. But all α and β are in one side of H, and so this is impossible.
QED

Any base must arise in this way, namely, there is a hyperplane not containing any roots such that the base is the set of indecomposible elements in one side of the hyperplane:

Take an element h in the dual cone $\{h \in E^*: (h, \alpha) > 0 \; \forall \alpha \in \Delta\}$. Then Δ and R^+ is contained in one side of $h^\perp$. R^- is contained in the other side.
Taking the indecomposable roots in the positive side of $h^\perp$ gives a base. This is Δ: both are base and hence have the same number of elements. $\alpha \in \Delta$ is indecomposable: suppose $\alpha = \beta + \gamma$ for $\beta, \gamma \in R^+$. Expressing as positive combinations of the base Δ, it forces β, γ are along the same direction of α and hence can only be α itself, impossible.

Prop. If Δ is a base for R, then $\Delta^\vee$ is a base for $R^\vee$.
Proof: From above Δ arises as indecomposible roots on one side of a hyperplane. $\alpha^\vee$ for $\alpha \in R^+$ lie on the same side, and that for $\alpha \in R^-$ lie on the other side. Thus indecomposible coroots on the positive side gives rise to a base $\Delta_0^\vee$ for $R^\vee$.
$\boldsymbol{\Delta}^\vee = \boldsymbol{\Delta}_0^\vee$: they have the same number of elements.
$H_\alpha \in \Delta^\vee$ is indecomposible: suppose $H_\alpha = H_\beta + H_\gamma$ for $\beta, \gamma \in R^+$. Expressing as positive combinations of the base $\Delta \ni \alpha$, it forces β, γ are along the same direction of α and hence can only be α itself, impossible.

Weyl chambers:
Connected components of $E - \bigcup_\alpha \alpha^\perp$.

Dominant (or fundamental) chamber C (relative to Δ):
$\langle \alpha, H \rangle > 0$ any $H \in C$ and $\alpha \in \Delta$.

{Base Δ} <-> {Weyl chamber *C*}:
-> take the dominant chamber relative to Δ.
It is the **dual cone** $\{H \in E^*: (\alpha, H) > 0 \; \forall \alpha \in \Delta\}$ which is a Weyl chamber since $(\alpha, H) \neq 0$ for any $\alpha \in R$.
<- Take any $H \in C$. $H^\perp$ does not contain any root and hence the indecomposible roots on the side $(\alpha, H) > 0$ define a base.
For all other elements $H' \in C$, since H' and H are in the same connected component of $E - \bigcup_\alpha \alpha^\perp$, (α, H_t) can never be zero and

hence cannot change sign for a path H_t connecting them.

Prop. Given a root, there exists a base containing it.
Proof: A base corresponds to a chamber. Given a root α, there is a chamber which has a facet given by $\alpha^\perp$ (and $(H,\alpha) > 0$ for H inside the chamber). $H^{>0}$ produces a base. α is indecomposible: We can take H very close to the hyperplane $\alpha^\perp$ such that (H,α) is minimal among all positive roots. QED

Prop. The Weyl group W is generated by s_α where $\alpha \in \Delta$.
It acts faithfully and transitively on the set of Weyl chambers.
(Hence as sets, $W \cong$ {Weyl chambers} $\cong$ {Bases}.)
Proof:
Let $W' \subset W$ be generated by s_α for $\alpha \in \Delta$.
Let C be the dominant chamber.
Want: for H' in any chamber, there is $w \in W'$ such that $w \cdot H' \in C$.
Suppose H' not in C. So there is a wall in between: there exists $\alpha \in \Delta$ such that $(\alpha, H') < 0$.
Reflection along this wall decreases the distance: Fix $H \in C$.

$$|H' - H|^2 - |s_\alpha \cdot H' - H|^2 = -\frac{4\langle\alpha, H'\rangle}{\langle\alpha,\alpha\rangle}\langle\alpha, H\rangle > 0.$$

Keep on doing this, gradually H' is reflected into C since W' is finite.
Hence W', and hence W, acts on Weyl chambers transitively.
Faithfulness is obvious.

For any root $\beta \in R, \beta \in \Delta_{C'}$ for some chamber C'. By above there exists some $w \in W'$ such that $w \cdot C' = C$. Then $w \cdot \beta \in \Delta$.
$s_\beta = w^{-1} \cdot s_{w\cdot\beta} \cdot w \in W'$.
Hence $W = W'$. QED

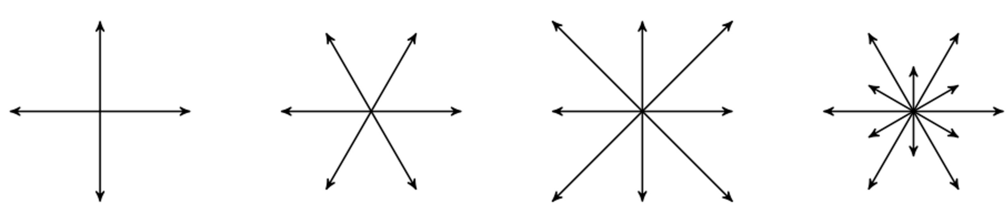
Minimal expression: Write $w \in W$ in a minimal product of reflections associated to elements in Δ.

Prop. Two distinct elements in $\overline{C}$ cannot lie in the same orbit of W.
Proof: Want to say $H' \neq w \cdot H$ for any w. Induction on length of minimal expression.
Let $1 \neq w = s_{\alpha_1} \dots s_{\alpha_k}$ be a minimal expression ($\alpha_i \in \Delta$).
Then C and $w \cdot C$ are on different sides of $\alpha_1^{\perp}$:
Again use induction. Suppose C and $s_{\alpha_1} \dots s_{\alpha_k} \cdot C$ are on the same side. So $s_{\alpha_1} \dots s_{\alpha_{k-1}} \cdot C$ is on another side by inductive assumption. Then $s_{\alpha_k} \cdot C$ and C are on different sides of $(u^{-1} \cdot \alpha_1)^{\perp}$ where $u = s_{\alpha_1} \dots s_{\alpha_{k-1}}$. But then $(u^{-1} \cdot \alpha_1)^{\perp} = \alpha_k^{\perp}$ and so $s_{\alpha_k} = s_{u^{-1} \cdot \alpha_1} = u^{-1} s_{\alpha_1} u$. Then $w = u \cdot s_{\alpha_k} = s_{\alpha_1} u = s_{\alpha_2} \dots s_{\alpha_{k-1}}$, contradicting the minimality.

Suppose $H' = w \cdot H$. Then $H' \in \alpha_1^{\perp}$. Thus $H' = s_{\alpha_2} \dots s_{\alpha_k} \cdot H$, contradicting the inductive assumption. QED

Prop: For $\alpha \in \Delta$, s_α preserves $R^+ - \{\alpha\}$.
Proof: Consider $\beta \in R^+ - \{\alpha\}$ and express it in terms of the base. It must involve an element γ in the base which is not α. $s_\alpha \cdot \beta = \beta - k\alpha$ and so it does not change the coefficient of γ, which is positive. Hence $s_\alpha \cdot \beta$ is still positive. QED

Dynkin diagram:
Vertices are base roots.
Number of edges between two vertices α, β is $\frac{\langle \alpha,\alpha \rangle}{\langle \beta,\beta \rangle}$ (WLOG $|\alpha| \geq |\beta|$) which is either 0,1,2,3. (**Recall that it determines the angle, which must be obtuse.**)
Direction of edge is from longer to shorter.
(Choice of base does not matter: any two are related by reflection.)



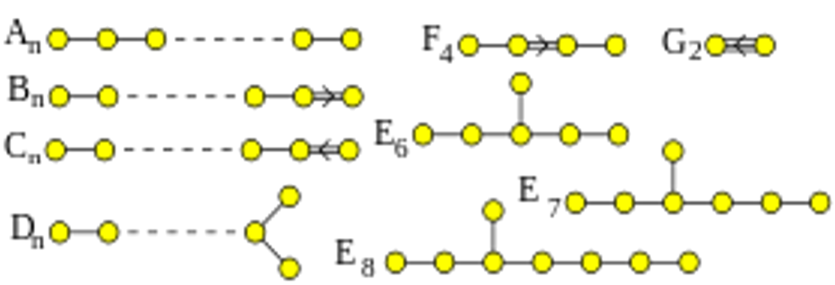
$A_1 \times A_1$ $\quad$ A_2 $\quad$ $B_2 \cong C_2$ $\quad$ G_2

Morphism of Dynkin diagram:
map between vertex sets preserving the numbers and directions of arrows between any two vertices.

R is irreducible <=> Dynkin diagram is connected:
<= $R = R_1 \cup R_2$, then $\Delta = \Delta_1 \cup \Delta_2$ which are orthogonal to each other. Then obviously the Dynkin diagram is disconnected.
=> If Dynkin disconnected, then $\Delta = \Delta_1 \cup \Delta_2$ which are orthogonal to each other. All roots are obtained from base by Weyl action. Since orthogonal the Weyl action preserves $E_i = \mathrm{Span}(\Delta_i)$. Hence any root is either in E_1 or E_2.

R_1 and R_2 are isomorphic <=> Dynkin diagrams are isomorphic:
WLOG assume irreducible.
=> Take base of R_1, mapping to a base of R_2. Then the isomorphism is an isometry up to scaling.
<= We have map between base roots, which is isometry up to scaling. Then it certainly respects Weyl group actions.

Classification:



Integral structure:
$E_{\mathbb{Z}}^* = \mathbb{Z} \cdot \{H_\alpha \in E^* \text{ for } \alpha \in \Delta\}$ gives the integral structure (which is a lattice in $E^* = \mathfrak{h}_{\mathbb{R}}$).
The dual is $E_{\mathbb{Z}} = \{\mu \in E : (\mu, H_\alpha) \in \mathbb{Z}\} = \mathrm{Hom}(E_{\mathbb{Z}}^*, \mathbb{Z})$. Recall

$$(\mu, H_\alpha) = \frac{2\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

So $R \subset E_{\mathbb{Z}}$.
The dual basis $\{H_\alpha^*\} \subset E_{\mathbb{Z}}$ of $\{H_\alpha\} \subset E_{\mathbb{Z}}^*$ is called the fundamental weights.
It is characterized by

$$(H_\alpha^*, H_\beta) = \frac{2\langle H_\alpha^*, \beta\rangle}{\langle \beta, \beta\rangle} = \delta_{\alpha\beta}.$$

A special element:

$$\boldsymbol{\delta} := \frac{\mathbf{1}}{\mathbf{2}} \sum_{\boldsymbol{\alpha} \in \boldsymbol{R}^+} \boldsymbol{\alpha}.$$

$(\delta, H_\alpha) = 1$ for all $\alpha \in \Delta$ (and hence $\delta \in E_{\mathbb{Z},+} = E_{\mathbb{Z}} \cap C$):

$$\frac{1}{2}(\alpha, H_\alpha) = 1.$$

For other $\beta \in R^+$, $s_\alpha \cdot \beta \in R^+$. If $\beta \perp \alpha$, then $\langle \beta, H_\alpha\rangle = 0$; if not, then $\beta \neq s_\alpha \cdot \beta$ and $\langle \beta + s_\alpha \cdot \beta, H_\alpha\rangle = 0$. Hence their contribution sum up to zero.

Partial ordering:
Like $\mathfrak{sl}(3, \mathbb{C})$, have partial ordering on E:
$\mu \geq \lambda$ if $\mu - \lambda \in \mathbb{R}_{\geq 0} \cdot \Delta$.

It has the following properties (proof skipped):

If $\boldsymbol{\mu} \in \overline{\boldsymbol{C}}$, then $\boldsymbol{\mu} \geq \mathbf{0}$. $\boldsymbol{w} \cdot \boldsymbol{\mu} \leq \boldsymbol{\mu}\ \forall \boldsymbol{w} \in \boldsymbol{W}$.
$\boldsymbol{\lambda} \in \mathbf{Conv}(\boldsymbol{W} \cdot \boldsymbol{\mu})$ if and only if $\boldsymbol{W} \cdot \boldsymbol{\lambda} \leq \boldsymbol{\mu}$.

If $\boldsymbol{\mu} \in \boldsymbol{E}_{\mathbb{Z},+} = \boldsymbol{E}_{\mathbb{Z}} \cap \boldsymbol{C}$, then $\boldsymbol{\mu} \geq \boldsymbol{\delta}$.

Exercises. (Section 8.12)

1. Let $\alpha, \beta \in R$ be linearly independent. If $\alpha + k\beta \in R$ for $k \in \mathbb{Z}_+$, then $\alpha + l\beta \in R$ for $l = 0, \dots, k$.
7. Suppose A is an isomorphism between two irreducible root systems. Show that it is a constant multiple of an isometry.
9. $P(H) := \prod_{\alpha \in R^+} \langle \alpha, H\rangle$.

Show that $P(w \cdot H) = \det(w)\, P(H)$ for all $w \in W$ and $H \in E$.

10. Show that if $-I \notin W$, then the Dynkin diagram must have a non-trivial automorphism.