Representation of semi-simple Lie algebra

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Main theorem: $\mathfrak{g}$ semi-simple.
{irreducible representations of $\mathfrak{g}$} <-> $\bar{C} \cap \mathfrak{h}_{\mathbb{Z}}^*$
where -> is given by taking the highest weight.

Weights μ are integral:
For any H_α for $\alpha \in \Delta$, restrict the representation to $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2, \mathbb{C})$ and so $(\mu, H_\alpha) \in \mathbb{Z}$.

The set of weights (and their multiplicities) is invariant under the Weyl group:
Recall for $\mathfrak{sl}(2, \mathbb{C})$ representation,
$U\pi(H)U^{-1} = -\pi(H)$ where $U = e^{\pi(X)}e^{-\pi(Y)}e^{\pi(X)}: V \to V$.
For $s_\alpha \in W$, consider $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2, \mathbb{C})$ and $U_\alpha: V \to V$.
For $v \in V_\mu$, $U_\alpha \cdot v \in V_{s_\alpha \cdot \mu}$:
$\pi(H_\alpha) \cdot (U_\alpha \cdot v) = U_\alpha \cdot (U_\alpha^{-1} \cdot \pi(H_\alpha) \cdot U_\alpha) \cdot v = U_\alpha \cdot (-\pi(H_\alpha)) v$
$= (\mu, s_\alpha \cdot H_\alpha)\, U_\alpha \cdot v$.
If $H \perp \alpha$, H commutes with X_α and Y_α and hence U_α. Then
$\pi(H) \cdot (U_\alpha \cdot v) = (\mu, H)\, U_\alpha \cdot v = (\mu, s_\alpha \cdot H)\, U_\alpha \cdot v$.
Hence $\pi(H) \cdot (U_\alpha \cdot v) = (\mu, s_\alpha \cdot H)\, U_\alpha \cdot v = (s_\alpha \cdot \mu, H)\, U_\alpha \cdot v$ for all H.

Proof of Main Theorem:
Exactly like $\mathfrak{sl}(3, \mathbb{C})$,
irreducible representations if and only if highest weight representations.
(=> acting by positive root vectors until reaching the highest.
<= semi-simple Lie algebra is completely reducible. Highest weight vector (which has multiplicity one) must belong to one of the irreducible factor, but it is cyclic and hence there is only one factor.)
Then take the highest weight, which is dominant integral: for H_α where $\alpha \in R^+$, consider $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2, \mathbb{C})$ and restrict as its representation. A highest weight vector v of π is in particular a highest weight vector of $\pi|_{\mathfrak{s}_\alpha}$. Hence $H_\alpha \cdot v = \lambda v$ where $\lambda \in \mathbb{Z}_{\geq 0}$.
Two highest weight representations with the same highest weight are isomorphic by the same proof. (Consider the subspace of $V \oplus W$ generated by the highest weight vectors (v, w) which is again a highest weight representation. The projection maps are isomorphisms.)

Now for $\mu \in \mathfrak{h}_{\mathbb{Z},+}^*$, need to cook up a μ-highest weight representation.
Cook up a canonical representation (called **Verma module**) which is infinite dimensional, and then take quotient! Do it below.
QED

Note: **highest weight infinite-dimensional representation may NOT be irreducible, and two with the same highest weight may not be isomorphic! Also the highest weight can be any complex numbers!**

$\mathfrak{sl}(2, \mathbb{C})$-Verma module W_μ with highest weight $\mu \in \mathbb{C}$:
Take formal span of $\{v_0, v_1, \ldots\}$. Define
$Y \cdot v_j = v_{j+1}$,
$H \cdot v_j = (\mu - 2j)v_j$,
$X \cdot v_0 = 0$,
$X \cdot v_j = X \cdot Y^j \cdot v_0$
$= Y \cdot X \cdot Y^{j-1} \cdot v_0 + H \cdot v_{j-1}$
$= Y \cdot X \cdot Y^{j-1} \cdot v_0 + (\mu - 2(j-1))v_{j-1} = \cdots = j(\mu - j(j-1))v_{j-1}$.

If $\mu = m \in \mathbb{Z}_{\geq 0}$,
$X \cdot v_{m+1} = 0$.
Thus $U_\mu = \mathrm{Span}\{v_{m+1}, \ldots\}$ is invariant. The quotient V_μ is then a finite dimensional highest weight representation.

For general $\mathfrak{g}$, a basis of W_μ is given by $Y_1^{k_1} \ldots Y_N^{k_N} \cdot v_0$ where v_0 is a highest weight vector and Y_i are all the negative roots in this order.

Enveloping algebra A:
An associative algebra with identity and $i: \mathfrak{g} \to A$ with $i([X,Y]) = i(X)i(Y) - i(Y)i(X)$, and $i(\mathfrak{g})$ generates A, that is, the smallest subalgebra with 1 containing $i(\mathfrak{g})$ is A.

Universal enveloping algebra $U_\mathfrak{g}$:
The enveloping algebra such that every other enveloping algebra is a quotient of $U_\mathfrak{g}$ (which is compatible with $i: \mathfrak{g} \to U_\mathfrak{g}$).

ex. $\mathfrak{g} = \mathbb{C}$. Then $U_\mathfrak{g}$ is the free algebra $\langle H \rangle$.

ex. $\mathfrak{sl}(2,\mathbb{C})$. $U_\mathfrak{g} = \langle X, Y, H \rangle / \langle XY - YX - H, HX - XH - 2X, HY - YH + 2Y \rangle$.
The above $\langle X, Y, H \rangle$ is the tensor algebra (which is free algebra quotient by bilinear relations).

Construction:
Take tensor algebra (treating $\mathfrak{g}$ as a vector space)
$$T(\mathfrak{g}) := \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k}.$$
Then define $U_\mathfrak{g}$ as the quotient by $\langle XY - YX - [X,Y] : X, Y \in \mathfrak{g} \rangle$.
It is universal:
If $i: \mathfrak{g} \to A$, then have $T(\mathfrak{g}) \to A$ (mapping 1 to 1_A) which is surjective. If $i([X,Y]) = i(X)i(Y) - i(Y)i(X)$, then the map descends to $U_\mathfrak{g} \to A$.

In particular,
representation $\mathfrak{g} \to A = \mathrm{End}(V)$ (where V can be infinite-dimensional) is one-to-one corresponding to $U_\mathfrak{g} \to \mathrm{End}(V)$.

Poincare-Birkhoff-Witt Theorem:
If $X_1, \ldots, X_N$ is a basis of $\mathfrak{g}$, then $i(X_1)^{n_1} \ldots i(X_N)^{n_N}$ form a basis of $U_\mathfrak{g}$.
(In particular $i: \mathfrak{g} \to U_\mathfrak{g}$ is injective.)

Proof: use induction on degree, skipped.

Corollary:
If $\mathfrak{h} \subset \mathfrak{g}$, then $U_\mathfrak{h} \subset U_\mathfrak{g}$.

Verma module with highest weight μ:
Want to make $1 \in U_\mathfrak{g}$ to be the highest weight vector.
Take the left ideal $U_\mathfrak{g} \cdot \langle H - (\mu, H) \text{ for } H \in \mathfrak{h}, X_\alpha \text{ for } \alpha \in R^+ \rangle$
(declaring 1 has weight μ, and declaring 1 is a highest weight vector.)
Verma module is the quotient vector space
$\boldsymbol{W_\mu := U_\mathfrak{g} / U_\mathfrak{g} \cdot \langle H - (\mu, H) \textbf{ for } H \in \mathfrak{h}, X_\alpha \textbf{ for } \alpha \in R^+ \rangle}$.
It loses the ring structure, but still has the left module structure of $U_\mathfrak{g}$. Thus it is a representation of $\mathfrak{g}$. Since $U_\mathfrak{g}$ is generated by $\mathfrak{g}$ (acting on 1), $[1] \in W_\mu$ is cyclic.

Need: $[1] \in W_\mu$ is non-zero. That is $1 \notin$ the left ideal in U_g.
Proof:

$\mathfrak{g}$ is complicated. Consider the following simpler subalgebras:

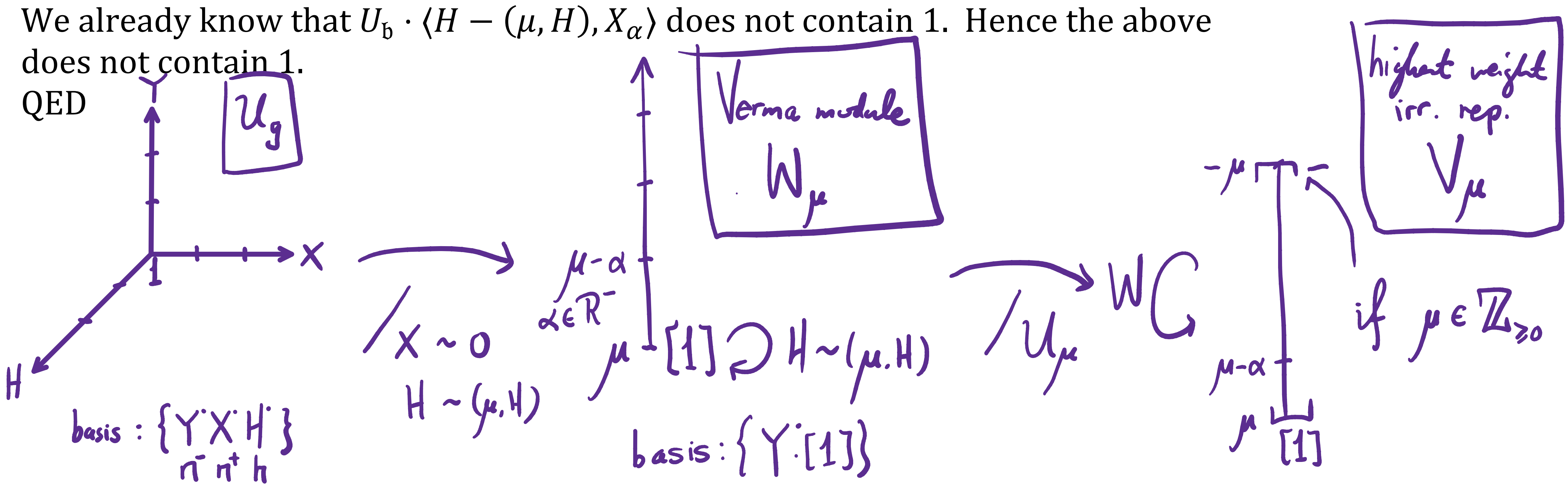
$$\mathfrak{n}^{\pm} := \bigoplus_{\alpha \in \mathrm{R}^{\pm}} \mathfrak{g}_{\alpha}\,.$$

$\mathfrak{b} := \mathfrak{n}^{+} \oplus \mathfrak{h}.$

They are subalgebras since $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subset \mathfrak{g}_{\alpha+\beta}$.

1 does not belong to the left ideal
$U_{\mathfrak{b}} \cdot \langle H - (\mu, H)$ for $H \in \mathfrak{h}$, positive root vectors $X\rangle$:
consider the 1d representation of $\mathfrak{b}$: $(X + H) \cdot v = (\mu, H)v$.
(It is a representation since $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{n}^{+}$ which acts by zero, and rank one representation is commutative. This is only valid for $\mathfrak{b}$ but not $\mathfrak{g}$.)
This corresponds to the morphism $U_{\mathfrak{b}} \to \mathbb{C}$. Kernel contains $U_{\mathfrak{b}} \cdot \langle H - (\mu, H), X_{\alpha}\rangle$.
$1_{U_{\mathfrak{b}}}$ maps to 1 and hence is not contained in kernel.

Now go back to $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^{-}$. By PBW theorem, basis of $\mathfrak{b}$ and $\mathfrak{n}^{-}$ induce a basis of $U_{\mathfrak{g}}$. Hence $U_{\mathfrak{g}} = \oplus Y_1^{n_1} \dots Y_N^{n_N} \cdot U_{\mathfrak{b}}$ where $\{Y_i : i = 1, \dots, N\}$ is a basis for $\mathfrak{n}^{-}$.
The ideal is
$\oplus Y_1^{n_1} \dots Y_N^{n_N} \cdot U_{\mathfrak{b}} \cdot (H - (\mu, H)) \oplus \oplus Y_1^{n_1} \dots Y_N^{n_N} \cdot U_{\mathfrak{b}} \cdot X_{\alpha}$
We already know that $U_{\mathfrak{b}} \cdot \langle H - (\mu, H), X_{\alpha}\rangle$ does not contain 1. Hence the above does not contain 1.
QED



For a basis $\{Y_i : i = 1, \dots, N\}$ of $\mathfrak{n}^{-}$, $Y_1^{k_1} \dots Y_N^{k_N} \cdot [1]$ form a basis of W_{μ}:
It is obvious that they generate since [1] is cyclic. They are linearly independent: for a linear combination of $Y_1^{k_1} \dots Y_N^{k_N}$ lying in the ideal of $U_{\mathfrak{g}}$, the coefficients must belong to both $\mathbb{C}$ and $U_{\mathfrak{b}} \cdot \langle H - (\mu, H), X_{\alpha}\rangle$, which is just $\{0\}$ since $U_{\mathfrak{b}} \cdot \langle H - (\mu, H), X_{\alpha}\rangle \not\ni 1$.

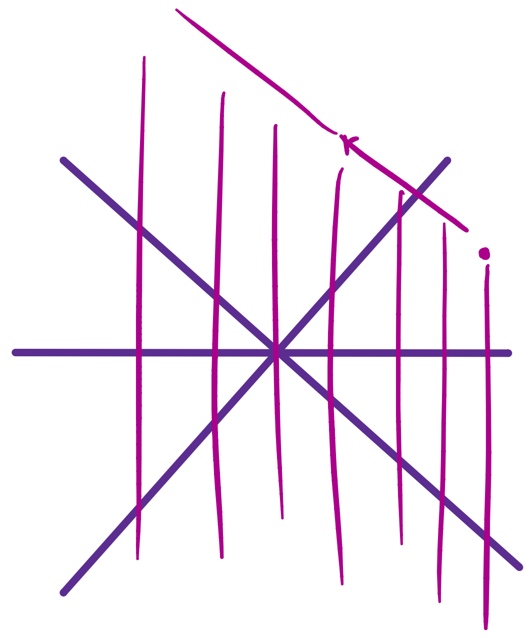
To make W_{μ} irreducible by quotient out certain invariant subspace.
Consider those vectors which can never get to [1] by X_{α} action:
$U_{\mu} = \{v \in W_{\mu} : X_1 \cdot \dots \cdot X_N \cdot v$ has no component in [1] for any $X_i \in \mathfrak{n}^{+}\}$.
(Use weight space decomposition of W_{μ} to talk about the components.)
U_{μ} is invariant: $X_1 \cdot \dots \cdot X_N \cdot Z \cdot v$ can be rearranged into $Y \dots H \dots X \dots \cdot v$.
$X \dots \cdot v$ has no component in [1], and action by $Y \dots H \dots$ keeps this property.
Then take $V_{\mu} := W_{\mu}/U_{\mu}$.

Prop. V_{μ} is irreducible.
Proof:
Consider invariant subspace S of W_{μ} containing but not equal to U_{μ}.
Let $v \in S - U_{\mu}$. So $u = X_1 \cdot \dots \cdot X_N \cdot v \in S$ has non-zero coefficient in [1]. Want to kill all other components of u.
Weight decomposition $u = a_0[1] + \sum_{\lambda \neq \mu} a_{\lambda} v_{\lambda}$. For $H \in \mathfrak{h}$,

$$H \cdot u = a_0\,(\mu, H)\,[1] + \sum_{\lambda \neq \mu} a_{\lambda}\,(\lambda\,, H)\,v_{\lambda}\,.$$

Then $(H - (\lambda\,, H) \cdot Id)$ kills the λ-component. The coefficient of [1] becomes

$a_0((\mu, H) - (\lambda, H))$,
and we choose H in the beginning to make sure $(\mu - \lambda, H) \neq 0$.
Keep on doing this, $[1] \in S$, hence S is the whole W_μ.



V_μ is infinite-dimensional in general.
Prop. For $\mu \in \bar{C} \cap \mathfrak{h}^*_{\mathbb{Z}}$, V_μ is finite-dimensional.
(In geometry, positivity and integrality corresponds to whether the Kaehler class comes from ample line bundle (polarization).)
Proof:
The set of weights (which are integral) of V_μ is invariant under Weyl group.
They are lower than the highest weight μ, and hence finitely many.
Each weight space is finite dimensional:
Recall $Y_1^{k_1} \dots Y_N^{k_N} \cdot [1]$ form a basis of W_μ for a basis $\{Y_i : i = 1, \dots, N\}$ of $\mathfrak{n}^-$.
Thus $Y_1^{k_1} \dots Y_N^{k_N} \cdot [1]$ spans the quotient V_μ.
There are just finitely many $Y_1^{k_1} \dots Y_N^{k_N} \cdot [1]$ with a given weight.

(If μ not integral, then don't have Weyl symmetry. Look at $\mathfrak{sl}(2, \mathbb{C})$.)
(Or consider **$\mathfrak{s}_\alpha$-action where $\alpha \in \Delta$**. We already know for $\mathfrak{sl}(2, \mathbb{C})$, V_μ constructed in this way is finite-dimensional.)

Need to construct W-action by using action of $e^{\pi(X_\alpha)}, e^{\pi(Y_\alpha)}$ for $\mathfrak{s}_\alpha = \langle X_\alpha, Y_\alpha, H_\alpha \rangle \subset \mathfrak{g}$.
Since V_μ may be infinite-dimensional, need to worry about **exponential operators on V_μ**. Once have $e^{\pi(X)}$, the same construction goes through for W-action. It is justified in the lemma below.
QED

Def. Locally nilpotent linear operator X on V:
for all $v \in V$, $X^k v = 0$ for some $k > 0$.
(For finite dimensions, this is same as nilpotent. Just take a basis.)

Have e^X for locally nilpotent X.

Lemma: For $\mathfrak{s}_\alpha = \langle X_\alpha, Y_\alpha, H_\alpha \rangle$ where $\alpha \in \Delta$, X_α, Y_α act locally nilpotently on V_μ if $\mu \in \bar{C} \cap \mathfrak{h}^*_{\mathbb{Z}}$.
Proof: Suffice to prove every $v \in V_\mu$ is contained in a finite-dimensional sub-representation of $\mathfrak{s}_\alpha$.
The set T of all such vectors form a vector space. It is **invariant** under whole $\mathfrak{g}$:
For $v \in T$, let $S \ni v$ be a finite-dimensional sub-reprsentation of $\mathfrak{s}_\alpha$. Take $\mathfrak{g} \cdot S$ which is still finite-dimensional. Note that $\mathfrak{g} \cdot S$ is NOT invariant under $\mathfrak{g}$, but it is invariant under $\mathfrak{s}_\alpha$. $X \cdot v \in \mathfrak{g} \cdot S$ for any $X \in \mathfrak{g}$, and hence $X \cdot v \in T$.
$T \neq \emptyset$, and hence $T = V_\mu$ as V_μ is irreducible:
Take the highest weight vector $[1] \in V_\mu$.
$(\mu, H_\alpha) \in \mathbb{Z}_{\geq 0}$ since $\mu \in \bar{C} \cap \mathfrak{h}^*_{\mathbb{Z}}$ and $\alpha \in \Delta$. Then $\{Y_\alpha^k \cdot [1] : k \in \mathbb{Z}\} \subset V_\mu$ forms a finite-dimensional sub-representation of $\mathfrak{s}_\alpha$. Hence $[1] \in T$.

Exercises. (Section 9.8)
1. Show that the Verma module W_μ is the maximal highest weight representation. Namely, for any highest weight representation V_μ, there is a surjective morphism $W_\mu \to V_\mu$ (and hence V_μ is a quotient of W_μ).
5. Let $\mu \in \mathfrak{h}^*$ and $R_+ = \{\alpha_1, \dots, \alpha_k\}$. Show that for the Verma module W_μ, the multiplicity of λ is the number of k-tuples $(n_1, \dots, n_k)$ such that $\lambda = \mu - n_1\alpha_1 - \dots - n_k\alpha_k$.