$D=[f_u \in \mathcal{M}^*(u): u \underset{open}{\subseteq} X,\ f_u/f_v \in \mathcal{O}^*(u\cap v)] \in \Gamma(\mathcal{M}^*/\mathcal{O}^*)$.

- Cartier divisor: locally defined by one meromorphic function -> Weil divisor
- Divisor line bundle $f_u e_u|_{u\cap v} = f_v e_v|_{u\cap v} \to$ mero. section s w/ $(s)=D$.
  $\mathcal{O}(D)|_u = \{h s : (h)+D|_u \geq 0\}$, $h \in \mathcal{M}_u$
- Picard group of line bundles
- Holomorphic bundle on C^n is trivial
- The exact sequence
- Canonical divisor $\sum_i D_i = (d\log z_1 \wedge \dots \wedge d\log z_n)$
- Toric Calabi-Yau $\sum_i D_i = (z^\nu) \Leftrightarrow (\nu, v_i)=1$.
  Holomorphic volume form $z^\nu\, d\log z_1 \wedge \dots \wedge d\log z_n$
- Polytope associated to toric divisor $P_D \triangleq \{\langle v_i, \cdot\rangle \geq -a_i\}$ for $D=\sum a_i D_i$.
- Global sections: $\Gamma(\mathcal{O}(D)) = \{(f)+D \geq 0\} = \mathrm{Span}\{z^\nu : \nu \in P_D \cap M\}$.
- Piecewise linear function associated to divisor $u(\sigma) \in M/\sigma^\perp$ defines lin. fcn on σ.
- Total space of divisor line bundle $\{\mathbb{R}_{\geq 0}\{(u(\sigma),0,-1)\}\} \triangleq \tilde{\Sigma}$ in $N_{\mathbb{R}} \times \mathbb{R}$.
- Globally generated <-> u is convex (Motivate from embedding)
- ample <-> u strictly convex
- Fano <-> reflexive polytope
- Ex. Compute the dimension of the section space of -K_X of X=P_2 and F_2

$\boxed{\text{Note: pole div. of } h_1+h_2 \leq \text{sum of that of } h_i}$

e.g. $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$.

$\mathrm{Div}_T(M) \to \mathrm{Pic}(X)$. (Gen. var.: $0 \to \Gamma(\mathcal{M}^*) \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0$.)

surjective: Consider toric charts u to trivialize L.

$e_u|_{(\mathbb{C}^*)^n} = f_u e_{(\mathbb{C}^*)^n}$ where f_u hol. & nowhere zero on $(\mathbb{C}^*)^n$

$\Rightarrow f_u$ is a monomial.

$\therefore \mathrm{Div}_T(M) \to \mathrm{Pic}(X) \to 0$.

Ker $= M$: If L is trivial, $\exists \{s_u$ hol. on $u\}$ s.t. $\{s_u e_u\}$ gives global nowhere-zero section.

$$s_u e_u|_{(\mathbb{C}^*)^n} = \underbrace{s_u|_{(\mathbb{C}^*)^n}}_{\text{hol. nowhere zero} \Rightarrow \text{monomial}} f_u e_{(\mathbb{C}^*)^n} = s_v|_{(\mathbb{C}^*)^n} f_v e_{(\mathbb{C}^*)^n}.$$

$\{f_u\}_u \sim \{s_u f_u\}_u$ def. global mero. fcn.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \mathrm{Div}_T(M) \arrow[r] \arrow[d, "\text{Weil}"] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "\wr\ \text{when smooth}"] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathbb{Z}\langle D_1 \dots D_m\rangle \arrow[r] & \mathbb{Z}\langle D_1 \dots D_m\rangle/\text{lin. equiv.} \arrow[r] & 0
\end{tikzcd}$$

Cartier

principal toric divisors

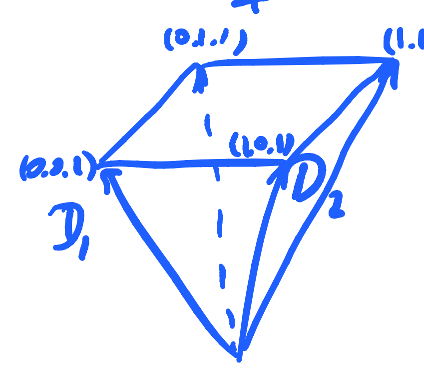
$\mathbb{Z}\langle D_1 \dots D_m\rangle \cong \mathbb{Z}^m = H^2(X,T)$, $\quad \mathbb{Z}\langle D_1 \dots D_m\rangle/\text{lin. equiv.} = H^2(X)$

dual sequence: $0 \to H_2(X) \to H_2(X,T) \to H_1(T) \to 0$, $\quad H_2(X,T) \cong \mathbb{Z}^m$, $\quad H_1(T) = N$.

toric Cartier div. $\longleftrightarrow \{u(\sigma) \in \overbrace{M}^{\text{monomial up to } \mathcal{O}^*/u_\sigma}/\sigma^\perp : \sigma \in \Sigma,\ u(\sigma) \mapsto u(\tau) \text{ for } \tau < \sigma\}$. $\longleftrightarrow$ piecewise lin. fcns supp. on Σ

Weil but not Cartier: conifold.

fan



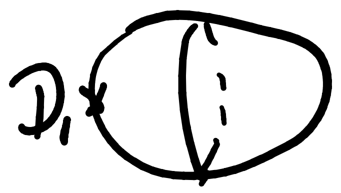
$\{XZ=YW\} \subset \mathbb{C}^4$.

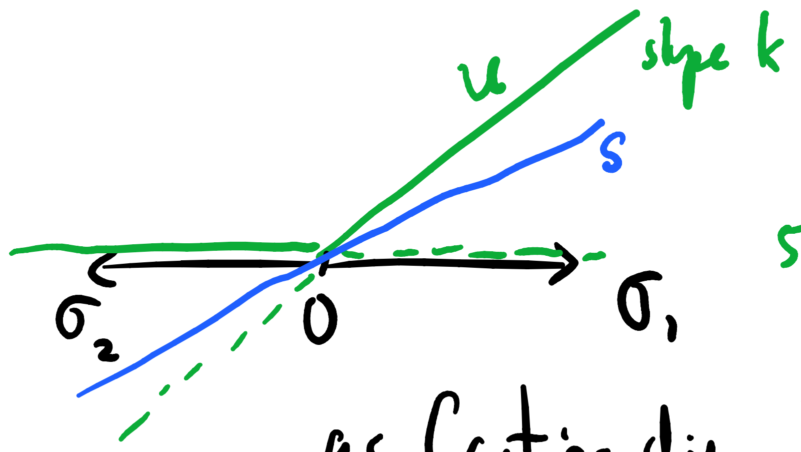
Impossible to define a toric irred. div. D_i by only 1 eqn.

e.g. $(z_1) = D_2 + D_3$. ← Weil but not Cartier

$(z_2) = D_3 + D_4$.

$(z_3) = D_1 + D_2 + D_3 + D_4$.

$\mathcal{O}_{\mathbb{P}^1}(kD_1)$ 
$\|$
e.g. $\mathcal{O}_{\mathbb{P}^1}(k)$. $\quad z_1^k e_1 = e_2$.



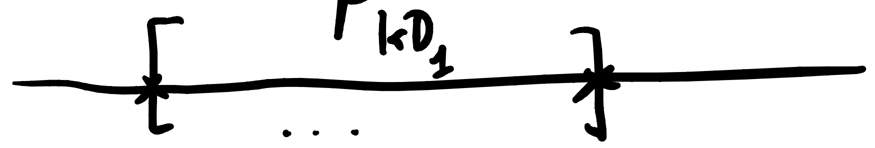
as Cartier div.

global sections: z^s $\quad z^s \cdot z^{u(\sigma)} e_\sigma$ hol.
$\Rightarrow -s \leq u$.

convex if $k \geq 0$.
strictly convex if $k > 0$.

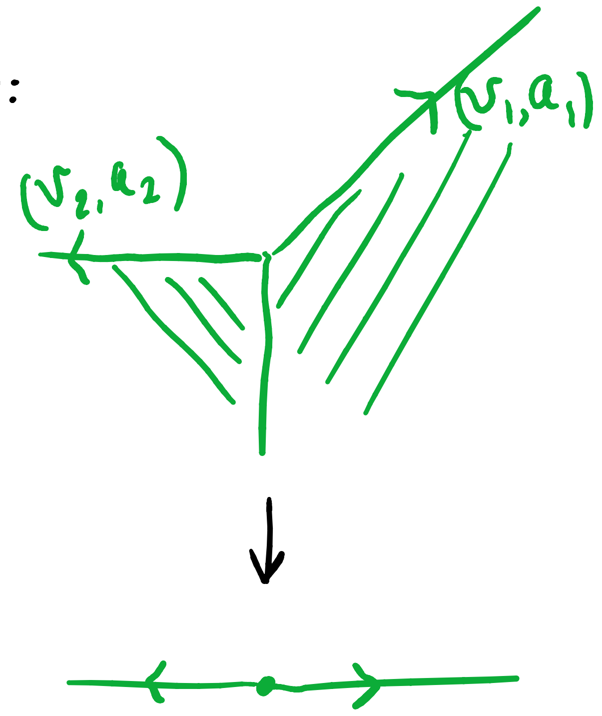
$= \phi$ when $k < 0$.

P_{kD_1}



$-k \quad\quad 0$
$\| \quad\quad \|$
$-u(\sigma_1) \quad -u(\sigma_2)$ as Weil div.

total space: $\mathcal{O}(kD_1)$



$z^{(0,-1)} = 1$ on $N_{\mathbb{C}}/N \simeq (\mathbb{C}^\times)^n$ gives a mero. section s with $(s) = -\sum_i ((0,-1),(v_i,a_i)) D_i$
$\uparrow$ fiber coord. over $N_{\mathbb{C}}/N$
$= \sum_i a_i D_i$.

$\boxed{\begin{array}{l}\mathcal{O}(D) \text{ global gen.} \\ \Leftrightarrow u \text{ convex}\end{array}}$: $\forall \sigma$ (max. cone), $\exists s \in M$ s.t. $\begin{cases} (s, v_i) \geq -a_i \ \forall i & \text{(i.e. } z^s \in \Gamma(\mathcal{O}(D))) \\ (s, v_i) = -a_i \text{ for } v_i \in \sigma. & \text{(i.e. } z^s \neq 0 \text{ on } U_\sigma) \end{cases}$

$\Updownarrow$

$\begin{cases} (-s, v_i) \leq a_i & \forall i \\ (-s, v_i) = a_i & \text{for } v_i \in \sigma. \end{cases}$

multiplicity of D_i in D
$\hookrightarrow a_i = (u(\sigma), v_i)$ $\forall v_i \in \sigma$ by def. $\Rightarrow -s = u(\sigma) \in M$. $(\sigma^\perp = 0.)$

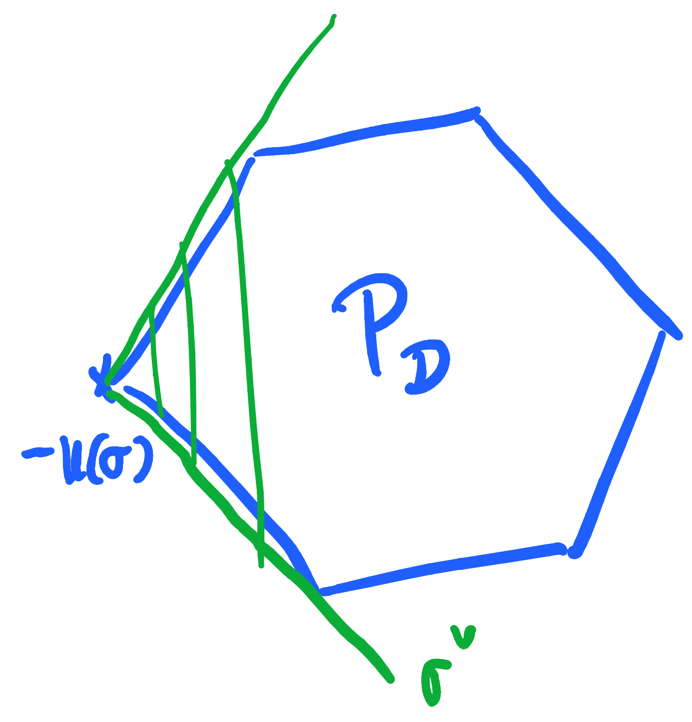
$a_i = (u(\langle v_i \rangle), v_i)$ $\forall i$ by def.
$\begin{array}{c}\forall \\ (u(\sigma), v_i)\end{array} \Longleftrightarrow$ convex. ($u \geq$ lin. part over every max. cone.)

$\boxed{\text{Kodaira embedding theorem: } L \text{ has metric } h \text{ such that } \mathrm{Ric}(h) \text{ defines a Käh. metric} \Rightarrow X \xhookrightarrow[|L|]{} \mathbb{P}^N}$

'a lot of sections'

$\underbrace{\mathcal{O}(D) \text{ very ample}} \Leftrightarrow \begin{cases} u \text{ is strictly convex} \\ \forall \text{ max cone } \sigma, \{u + u(\sigma) : u \in P_D \cap M\} \text{ generates } \sigma^\vee \cap M. \end{cases}$

$\Big\downarrow$ is embedding



Pf: $X_\Sigma \xrightarrow{[z^\nu : \nu \in P_D]} \mathbb{P}^N$. (globally gen. $\Rightarrow$ well-def.)

u strictly convex $\Leftrightarrow (\underset{\text{max cone}}{u(\sigma)}, v_i) < a_i \ \forall v_i \notin \sigma$

i.e. $z^{-u(\sigma)} = 0$ on $X_\Sigma - U_\sigma$

$$\begin{array}{ccc} \therefore \ U_\sigma & \xrightarrow{\left(\frac{z^u}{z^{-u(\sigma)}} : u \in P_D\right)} & \mathbb{C}^N = \{z^{-u(\sigma)} \neq 0\} \\ \cap & & \cap \\ X_\Sigma & \longrightarrow & \mathbb{P}^N \end{array}$$

(Hence $p \in U_\sigma$ & $q \in X - U_\sigma$ have different values)

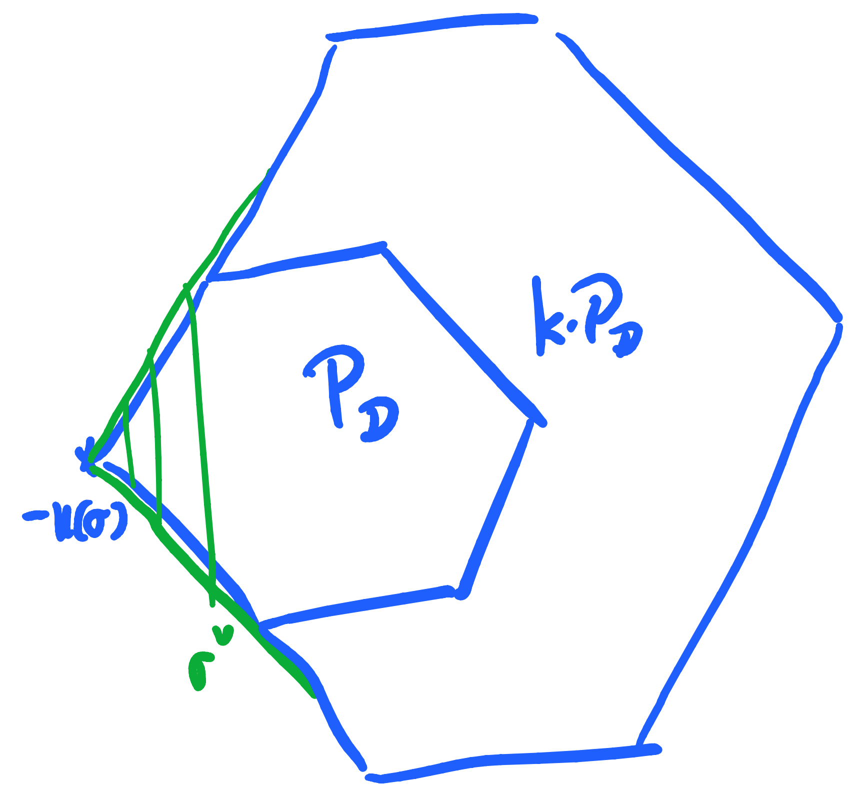
$(z^{u+u(\sigma)})$ is embedding

$\Updownarrow$

$\{u + u(\sigma) : u \in P_D\}$ generates $\sigma^\vee \cap M$. $\quad \mathbb{C}[u + u(\sigma)]$

$\mathcal{O}(D)$ ample $\Leftrightarrow$ u is strictly convex.

($\Leftrightarrow \mathcal{O}(kD)$ is very ample for $k \gg 0$.)



Pf: $\underbrace{\sigma^\vee \cap M}_{\text{fin. gen.}}$ gen. by $\underbrace{(k \cdot P_D)}_{P_{kD}} \cap M + u(\sigma)$.