$$H^0(X_\Sigma, \mathcal{O}(D)) = \underbrace{\bigoplus_{\mu\in M} H^0(X, \mathcal{O}(D))_\mu}$$

$$= \begin{cases} \mathbb{C}\cdot z^\mu & \text{if } \mu\in P_D \\ 0 & \text{otherwise} \end{cases}$$

$$= H^0_c(|\Sigma|, A(\mu)) \qquad \{p : z^\mu(p) = \infty\}$$

where

$$A(\underset{M}{\overset{\in}{\mu}}) \triangleq \{v\in |\Sigma| : \mu(v) > u_D(v)\} \subset |\Sigma|.$$

Thm.: $H^i(X, \mathcal{O}(D)) = \bigoplus_{\mu\in M} H^i_c(|\Sigma|, A(\mu))$.

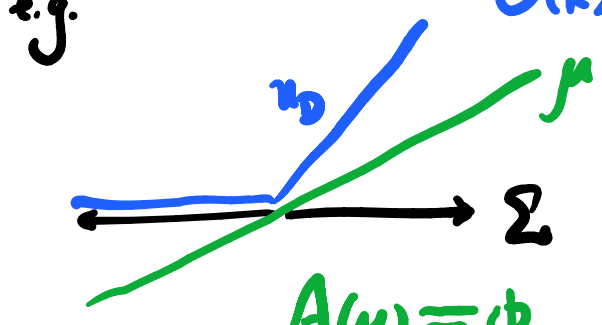
Cor.: $H^{>0}(X, \mathcal{O}(D)) = 0$ if $\mathcal{O}(D)$ is globally gen. and $|\Sigma|$ convex.

Pf: u_D convex $\Rightarrow$ $u_D - \mu$ convex $\forall \mu$

$\Rightarrow A(\mu) = \{u_D - \mu < 0\}$ convex $\forall\mu$.

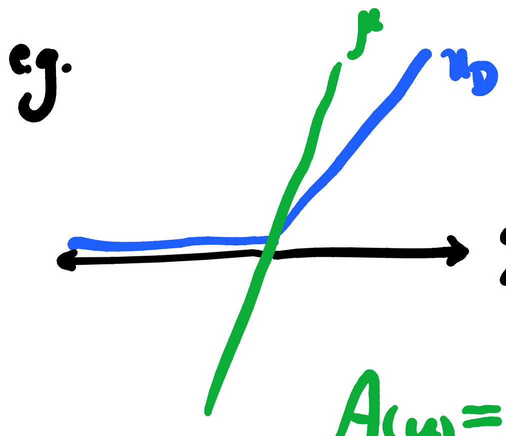
$\therefore H^{>0}_c(|\Sigma|, A(\mu)) = 0$. $\ast$

e.g.



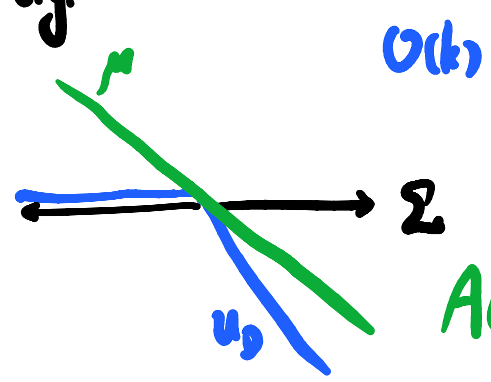
$$H^i_c(\mathbb{R}, \phi) = \begin{cases} 0 & i=0 \\ \mathbb{C} & i=1 \end{cases}$$

e.g.



$$H^\cdot_c(\mathbb{R}, \mathbb{R}_{>0}) = 0.$$

e.g.



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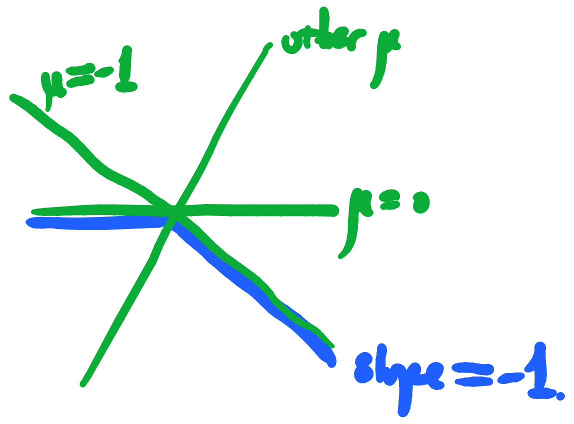
e.g. $\mathcal{O}_{\mathbb{P}^1}(-1)$.

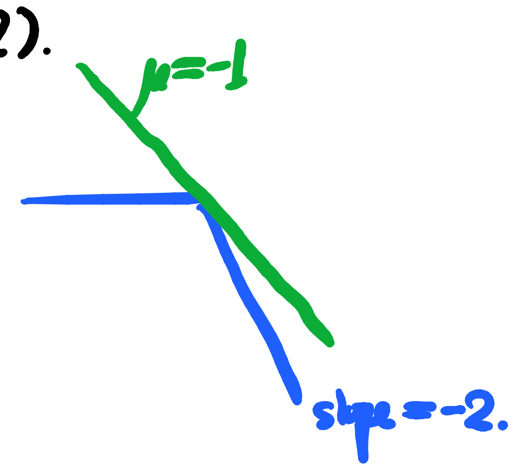


$$A(\mu) = \begin{cases} \mathbb{R}_{>0} & \mu\geq 0 \\ \mathbb{R}_{<0} & \mu\leq -1. \end{cases}$$

$H^\cdot(\mathbb{R}, A(\mu)) = 0$.

$\therefore H^\cdot(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$.

e.g. $\mathcal{O}_{\mathbb{P}^1}(-2)$.



$A(\mu=-1) = \mathbb{R}-\{0\}$.

$$H^k(\mathbb{R}, A(\mu=-1)) = \begin{cases} \mathbb{C} & k=1 \\ 0 & \text{otherwise}. \end{cases}$$

$H^\cdot(\mathbb{R}, A(\mu)) = 0$ for other μ.

$$\therefore H^k(\mathcal{O}_{\mathbb{P}^1}(-2)) = \begin{cases} \mathbb{C} & k=-1 \\ 0 & \text{otherwise} \end{cases}$$

Why thm. holds: compute by Čech coho. for toric charts.

$$\check{C}^p(X, \mathcal{O}(D)) = \bigoplus_{\sigma_0 \dots \sigma_p} H^0(U_{\sigma_0 \cap \dots \cap \sigma_p}, \mathcal{O}(D))$$

$$H^i_{\mathbb{C}}(\mathbb{R}, \mathbb{R}\text{-}\{a\}) = \begin{cases} 0 & i=0 \\ \mathbb{C} & i=1 \end{cases}$$

$$= \bigoplus_{\mu \in M} \underbrace{\bigoplus_{\sigma_0 \dots \sigma_p} H^0_{\mathbb{C}}(|\sigma_0 \cap \dots \cap \sigma_p|, A(\mu) \cap |\sigma_0 \cap \dots \cap \sigma_p|)}_{\check{C}^p_{\mathbb{C}}(|\Sigma|, A(\mu))}$$

($\mathbb{C}$-loc. const. fns supp. away from $A(\mu)$)

Note: $H^{>0}_{\mathbb{C}}(|\sigma_0 \cap \dots \cap \sigma_p|, \underbrace{A(\mu) \cap |\sigma_0 \cap \dots \cap \sigma_p|}) = 0$

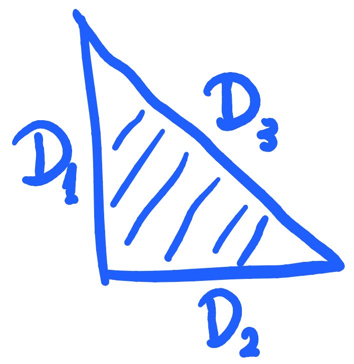
convex since $u_D|_{|\sigma_0 \cap \dots \cap \sigma_p|}$ is lin.

$$\therefore \ \chi(\mathcal{O}(D)) = h^0(\mathcal{O}(D)) = |P_D \cap M|.$$

Riemann-Roch thm.:

$$\chi(\mathcal{O}(D)) = (\underbrace{ch(\mathcal{O}(D))}_{\exp(D)}, \underbrace{Td_X}_{\prod_{i=1}^{m} \frac{D_i}{1-\exp(-D_i)}})$$

$\boxed{\text{purely combinatorial.}}$

e.g. 

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x+y \leq k \end{cases}$$ corr. to $\mathcal{O}(kD_3)$.

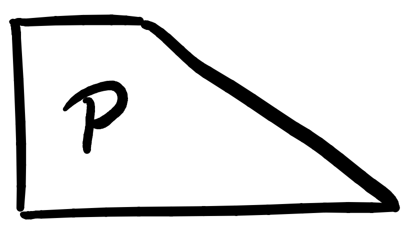
$D_1 = D_2 = D_3 = D$.

$$\chi = \underbrace{\exp(kD)}_{1+kD+\frac{k^2}{2}pt} \cdot \underbrace{\left(\frac{D}{1-\exp(-D)}\right)^3}$$

$$\frac{1}{1-\frac{D}{2}+\frac{D^2}{6}} = 1+\frac{D}{2}-\frac{D^2}{6}+\frac{D^2}{4}$$
$$= 1+\frac{D}{2}+\frac{pt}{12}$$

$$= \underbrace{\frac{3}{4}+\frac{1}{4}}_{1}+\frac{3k}{2}+\frac{k^2}{2} = \frac{(k+1)(k+2)}{2}$$ #

ex. Compute $|P \cap D|$ where $\underset{\text{polytope}}{P \subset} \mathbb{R}^2$ is def. by
$x \geq 0$;
$y \geq 0$;
$y \leq 1000$;
$x+2y \leq 1000$.



Characteristic classes $\quad$ local frame: $\overbrace{d\log z_1 \dots d\log z_k, dz_{k+1} \dots dz_n}$ around $D_1 \cap \dots \cap D_k - \bigcup_{i=k+1}^{m} D_i$

Prop. $0 \to \Omega^1_X \to \Omega^1(\log D) \xrightarrow[\mathrm{Res}]{} \bigoplus_{i=1}^{m} \mathcal{O}_{D_i} \to 0$ is exact. ($\mathrm{Res}=0 \Leftrightarrow$ no simple pole)

$$\sum_i f_i \, dz_i \longmapsto \sum_{i=1}^{k} (z_i f_i) \, d\log z_i + \sum_{i=k+1} f_i \, dz_i$$

Moreover $\Omega^1(\log D) \simeq \underset{\text{trivial}}{\underline{\mathbb{C}^n}}$ $\quad$ global frame: $d\log z^{\nu_1} \dots d\log z^{\nu_n}$ for a basis $\{\nu_1, \dots \nu_n\}$ of M.

Hence $c(\Omega^1_X) = \prod_{i=1}^{m} \underbrace{c(\mathcal{O}_{D_i})^{-1}}_{1-D_i}$ $\quad \left(\begin{array}{c} 0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0 \\ \Rightarrow c(\mathcal{O}_D)^{-1} = (1-D_i). \end{array} \right)$

$$c(T_X) = \prod_{i=1}^{m} (1+D_i) \quad (c_i(E^*) = (-1)^i c_i(E) \text{ by using Chern roots})$$

$$= \sum_{\sigma \in \Sigma} V_\sigma .$$

$$Td(\Omega^1_X) = \prod_{i=1}^{m} \underbrace{Td(\mathcal{O}_{D_i})^{-1}}_{Td(\mathcal{O}(-D_i))}$$

$$Td(T_X) = \prod_{i=1}^{m} Td(\mathcal{O}(D_i)) = \prod_{i=1}^{m} \frac{D_i}{1-\exp(-D_i)} . \quad \#$$