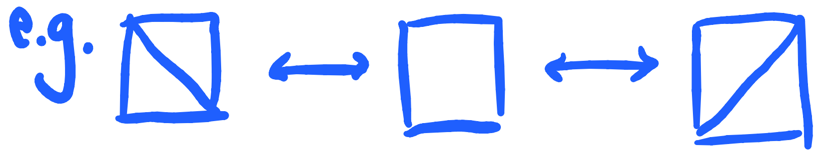
Wednesday, January 6, 2016 9:32 AM

- Motivation: Give a more global way to understand toric. Need to choose "unstable" strata to delete. Important for moduli theory.
- The exact sequence $0\to k\to\mathbb{Z}^m\to N\to 0 \rightsquigarrow N_{\mathbb{C}}/N=(\mathbb{C}^\times)^m/K$. $(K=k_{\mathbb{C}}/k\simeq(\mathbb{C}^\times)^r)$
- Naïve quotient is not even Hausdorff. Eg C^m/C^*
- Geometric quotient C^m/K: identify two orbits if their closures intersect. Eg C^m//C^* = {0}
  Algebraically: take G-invariant ring $\mathrm{Spec}(\mathbb{C}[z_1,\dots,z_m])^K)$.
- GIT: choice of a character -> G-equiv. line bundle $\chi\in k^*=\mathrm{Hom}(K,\mathbb{C}^\times)\Rightarrow K\curvearrowright \underset{\downarrow\ \mathbb{C}^r}{\mathbb{C}}\ \mathcal{L}_\chi$ $g(p,t)=(g\cdot p,\chi(g)\cdot t)\Rightarrow$ l.b. on $\mathbb{C}^m/\!/K$.
  ↑called polarization
- Stable, semi-stable and unstable points
- Affinization; residual action
- From character to the polytope and normal fan
- Description of unstable points in terms of the polytope or fan
- Secondary fan: classify different fans containing the same set of rays

e.g. $\mathbb{C}^2/\mathbb{Z}_2\leftrightarrow\mathcal{O}_{\mathbb{P}^1}(-1)$.

e.g. 

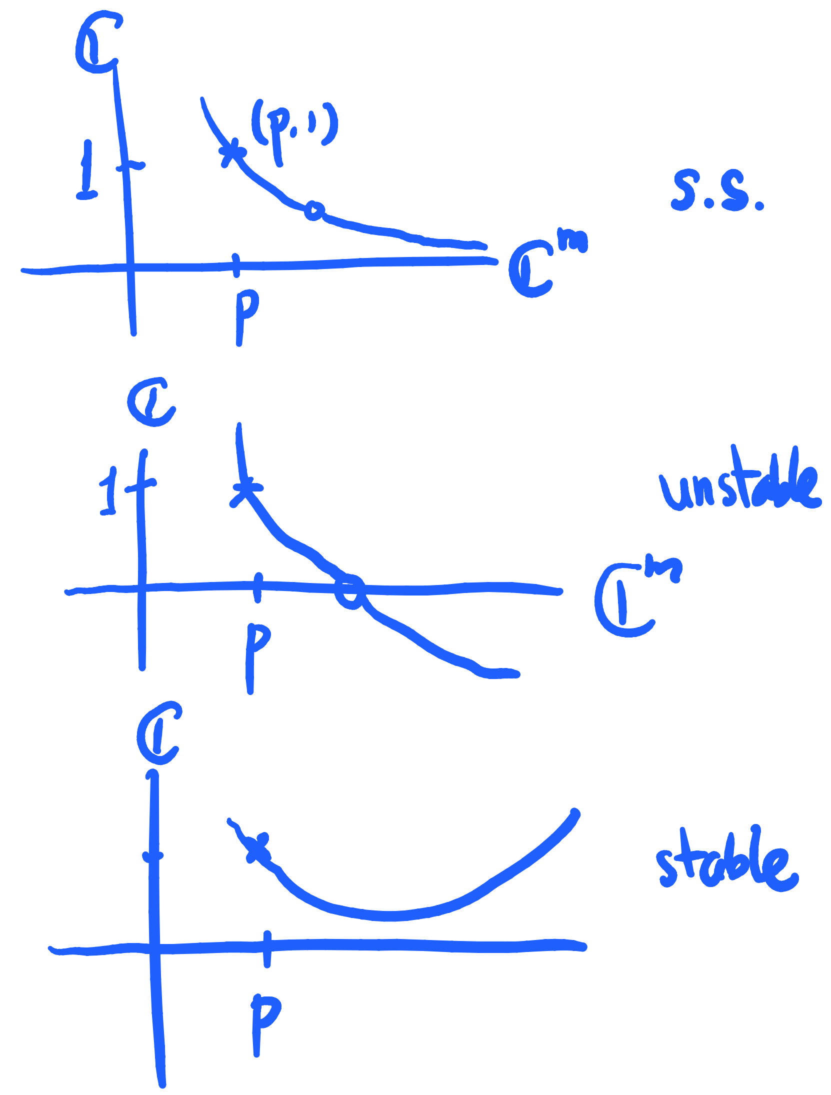
$\langle s,\zeta\rangle$

Replace $\mathbb{C}[z_1\dots z_m]^K$ by $\Gamma(\mathbb{C}^r,\mathcal{L}_\chi)^K=\{s(g\cdot x)=\chi(g)s(x)\}=\mathcal{O}(\mathcal{L}_\chi^{-1})^K$

$s\cdot\zeta^k\in\mathcal{O}(\underline{\mathbb{C}})^K$

- $p\in\mathbb{C}^m$ s.s. if $\overline{K\cdot(p,1)}\cap 0_{\text{sect}}=\emptyset\iff\exists s\in\Gamma(\mathbb{C}^m,\mathcal{L}_{\chi^k})^K$ $k>0$ s.t. $s(p)=1$.
  unstable o/w.
  polystable if $K\cdot(p,1)$ is closed $\iff K\curvearrowright\{s\neq0\}$ is closed.
  stable if further K_p finite.

(⇐) $\{s\zeta^k=1\}\supset\overline{K\cdot(p,1)}$ never intersect $\{\zeta=0\}$.
(⇒) $\exists$ a K-inv. poly. $f=\sum_{k=0}^{\infty}s_k\zeta^k$ on $\underline{\mathbb{C}}$ s.t. $(p,1)\in\{f=1\}$ and $\{f=1\}\cap 0_{\text{sect}}=\emptyset$.
Then $s_0(p)\neq1$, and so $\exists\, s_k(p)\neq0$ for $k>0$.



- Mumford criterion: use 1-parameter subgp to test.
  s.s. if $\forall$ 1-para. subgp $\lambda:\mathbb{C}^\times\to K$, $\lim_{t\to0}\lambda(t)\cdot(p,1)\notin 0_{\text{sect}}$.
  unstable o/w.
  polystable if $\lim_{t\to0}\lambda(t)\cdot(p,1)\in K\cdot(p,1)$ (if exists).
  stable if $\lim_{t\to0}\lambda(t)\cdot(p,1)$ exists $\Rightarrow\lambda\equiv1$.

- $\mathbb{C}^m/\!/_\chi K=\mathrm{Proj}\left(\bigoplus_{d=0}^{\infty}\Gamma(\mathbb{C}^m,\mathcal{L}_{\chi^d})^K\right)\simeq(\mathbb{C}^m)^{ss}_\chi/\!/K\xrightarrow[\text{affinization}]{}\mathbb{C}^m/\!/K\simeq\mathrm{Spec}\,\Gamma(\mathbb{C}^m,\mathcal{O})^K$. e.g. $\mathbb{C}^3/\!/_{\chi=1}\mathbb{C}^\times\simeq\mathbb{P}^2\longrightarrow\mathbb{C}^3/\!/\mathbb{C}^\times=\{0\}$.

· $0 \to M \to \mathbb{Z}^m \xrightarrow{\gamma} k^* \to 0$. (dual sequence)
$\quad\quad\quad\quad\quad\quad\quad\quad \underset{\chi}{\Psi}$

$\gamma(a) = \chi$; different choices related by translation.

$P_\chi \doteq \gamma_{\mathbb{R}}^{-1}(\chi) \cap \mathbb{R}^m_{\geq 0}$. $F_i = P_\chi \cap (i^{th}$ coord. hyperplane of $\mathbb{R}^m)$. $P_\chi - a \subset M_{\mathbb{R}} = \operatorname{Ker} \gamma_{\mathbb{R}}$.

$\rightsquigarrow \Sigma$ normal fan.

intuitively $z_i = 0$ corr. to the face F_i.

Prop.: $\underset{(z_1, \dots z_m)}{\overset{z}{\shortparallel}} \in \mathbb{C}^m$ is unstable wrt $\chi \iff \bigcap_{i: z_i = 0} F_i = \phi \iff \mathbb{R}_{\geq 0}\{v_i : z_i = 0\}$ not cone of Σ.

$(\iff \mathbb{R}_{\geq 0}\{e_i : z_i = 0\} / g_{\mathbb{R}}$ not cone of Σ.)

Pf: $\quad \underline{z}$ is s.s. $\iff \exists \overset{\text{h.d.}}{F}$ w/ $F(g \cdot z) = \chi^d(g) F(z)$ and $F(\underline{z}) \neq 0$.

$\iff \exists^{\text{hd}}$ monomial $z^{\vec{b}}$ w/ $\underset{\gamma(\vec{b})(g) \cdot z^{\vec{b}}}{\overset{g^* z^{\vec{b}}}{\shortparallel}} = \chi^d(g) \cdot z^{\vec{b}}$ and $z^{\vec{b}} \neq 0$

$\iff \exists \vec{b} \in d \cdot P_\chi$ and $\underbrace{\underline{z}^{\vec{b}} \neq 0}_{b_i = 0 \ \forall i \text{ w/ } z_i = 0}$

$\iff \vec{b}/d \in \bigcap_{i: z_i = 0} F_i$

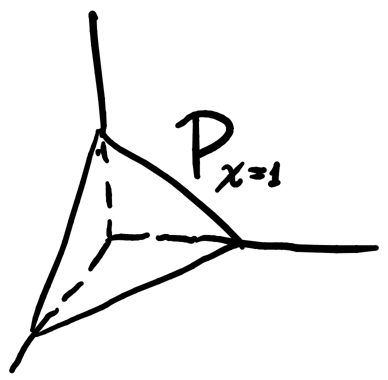
$$\boxed{\begin{array}{l} \mathbb{R}_{\geq 0}\{e_i : i \in I\} \text{ is always a cone in } \mathbb{R}^m_{\geq 0}; \\ \mathbb{R}_{\geq 0}\{v_i : i \in I\} \subset N_{\mathbb{R}} \text{ may not.} \end{array}}$$

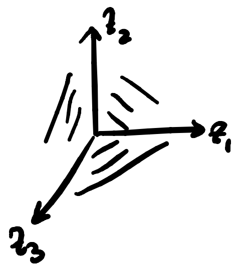
· $X = (\mathbb{C}^m - \underset{\text{set of unstable points}}{Z_\chi}) /\!/ G$, where $Z_\chi = \bigcup_{\substack{I: \\ \bigcap_{i \in I} F_i = \phi}} \{z_i = 0 : i \in I\} = \bigcup_{\substack{I: \\ \mathbb{R}_{\geq 0}\{v_i : i \in I\} \text{ not a cone of } \Sigma}} \{z_i = 0 : i \in I\} = \bigcup_{\substack{\text{prim. collection} \\ I}} \{z_i = 0 : i \in I\}$

$\mathbb{R}_{\geq 0}\{v_i : i \in I\}$ not a cone of Σ and $\mathbb{R}_{\geq 0}\{v_i : i \in I - \{j\}\}$ is a cone of Σ $\forall j \in I$.

· $X = \phi$ if $\chi \notin \underbrace{\gamma_{\mathbb{R}}(\mathbb{R}^m_{\geq 0})}_{\text{cone of effective divisors}}$.

e.g. $\mathbb{P}^2$.





$Z = \{z_1 = z_2 = z_3 = 0\}$.

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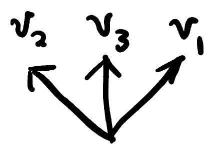
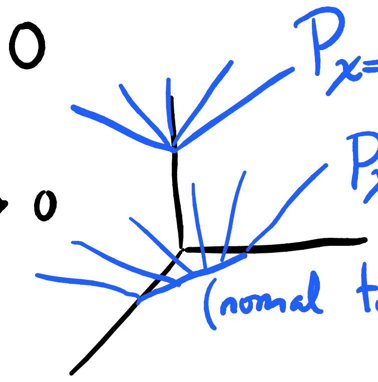
e.g. $K_{\mathbb{P}^1} \longleftrightarrow \mathbb{C}^2/\mathbb{Z}_2$.



$$0 \to \underset{k}{\mathbb{Z}} \xrightarrow{\begin{pmatrix}1\\1\\-2\end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix}1&-1&0\\1&1&1\end{pmatrix}} \underset{N}{\mathbb{Z}^2} \to 0$$

$$0 \to M \xrightarrow{\begin{pmatrix}1&1\\-1&1\\0&1\end{pmatrix}} \mathbb{Z}^3 \xrightarrow{(1\ 1\ -2)} k^* \to 0$$



$P_{\chi=-1} \Rightarrow \mathbb{C}^2/\mathbb{Z}_2$. $Z = \{z_3 = 0\}$

$P_{\chi=1} \Rightarrow K_{\mathbb{P}^1}$. $Z = \{z_1 = z_2 = z_3 = 0\}$

(normal to $(1,1,-2)$)

affinization: $K_{\mathbb{P}^1} \to \mathbb{C}^2/\mathbb{Z}_2$.

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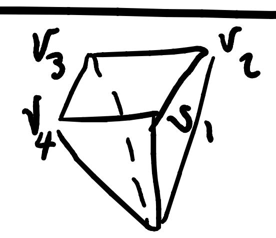
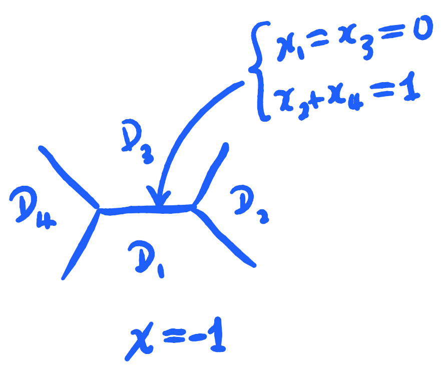
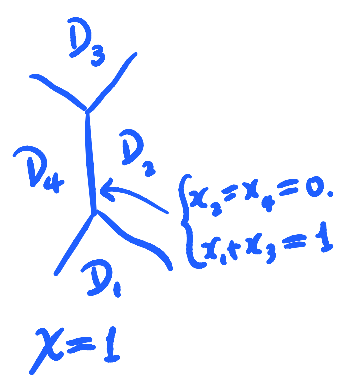
e.g. $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.



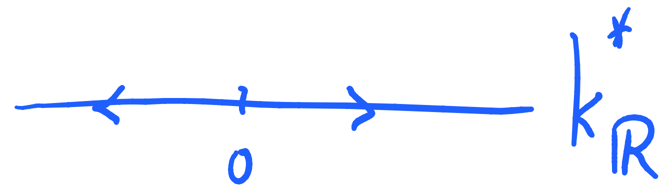
$$0 \to \underset{k}{\mathbb{Z}} \xrightarrow{\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}} \mathbb{Z}^4 \to \underset{N}{\mathbb{Z}^3} \to 0.$$

$$0 \to M \to \mathbb{Z}^4 \xrightarrow{(1\ -1\ 1\ -1)} k^* \to 0.$$



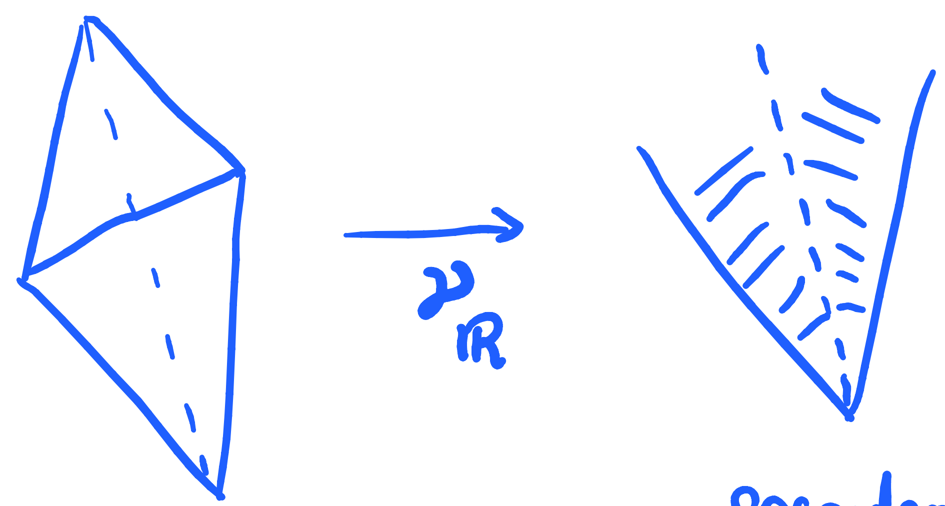


· In all these examples, have geometric transition at $\chi = 0$.



· $\gamma_{\mathbb{R}}$(coord. cones $\subset \mathbb{R}^m_{\geq 0}$) divide $\gamma_{\mathbb{R}}(\mathbb{R}^m_{\geq 0})$ into chambers $\longrightarrow$ secondary fan

some div. are blown down



secondary fan