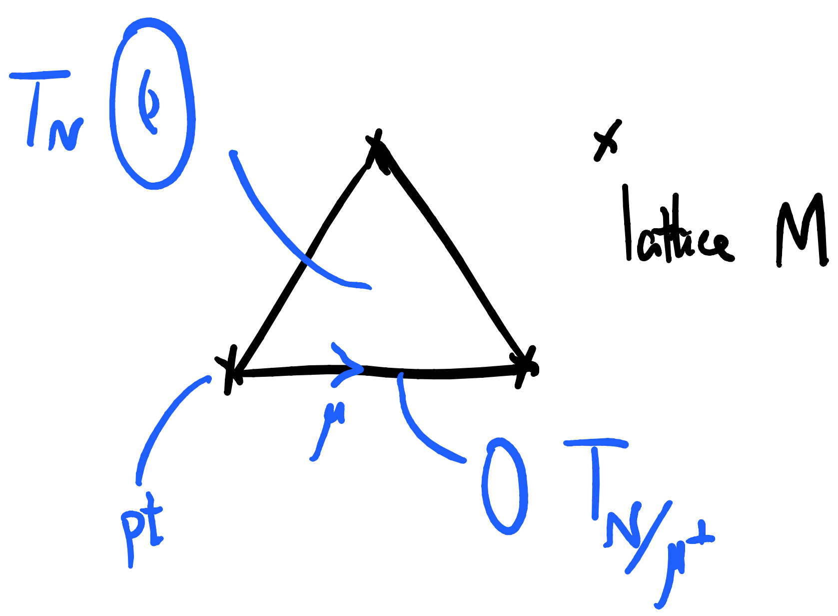
- Motivation: endow a Kaehler structure; better understand the role of the polytope. Outcome: toric manifold is Kaehler
- Def. of Kaehler manifold. Holonomy: U(n). (X, J, g, ω): $g(u, Jv) = \omega(u,v)$, $g(Ju, Jv) = g(u,v)$, $d\omega = 0$.
- Example: C^n = T^* R^n (also P^n and its submanifolds)
- Newton's equation and Hamiltonian mechanics on T^* R^n
- T^n action on C^n
- Definition of moment map.
- Eg. C^n -> R^n_{>=0} => Lagrangian torus fibration
- The exact sequence and corresponding torus action
- Moment map of subgroup action
- Naïve quotient is not symplectic: dimension is not correct
- Symplectic quotient $\mathbb{C}^N/\!/_{\chi} K$
- Symplectic structure descends to quotient
- Eg. P^2 , K_P^1
- Residual torus action and moment map
- Kempf-Ness theorem: symplectic quotient = GIT quotient

$$\begin{tikzcd}
 & M_{\mathbb{R}}+a \arrow[d, phantom, "\wr"] & \\
T_N \curvearrowright X \arrow[r] \arrow[d, equal] \arrow[dr, "\mu"] & \nu^{-1}\{\chi\} \subset \mathbb{R}^m_{\geq 0} \arrow[d, "\nu"] & \ni a \arrow[d] \\
\mu^{-1}\{\chi\}/\!/K & \mathbb{R}^{m-n} & \ni \chi
\end{tikzcd}$$

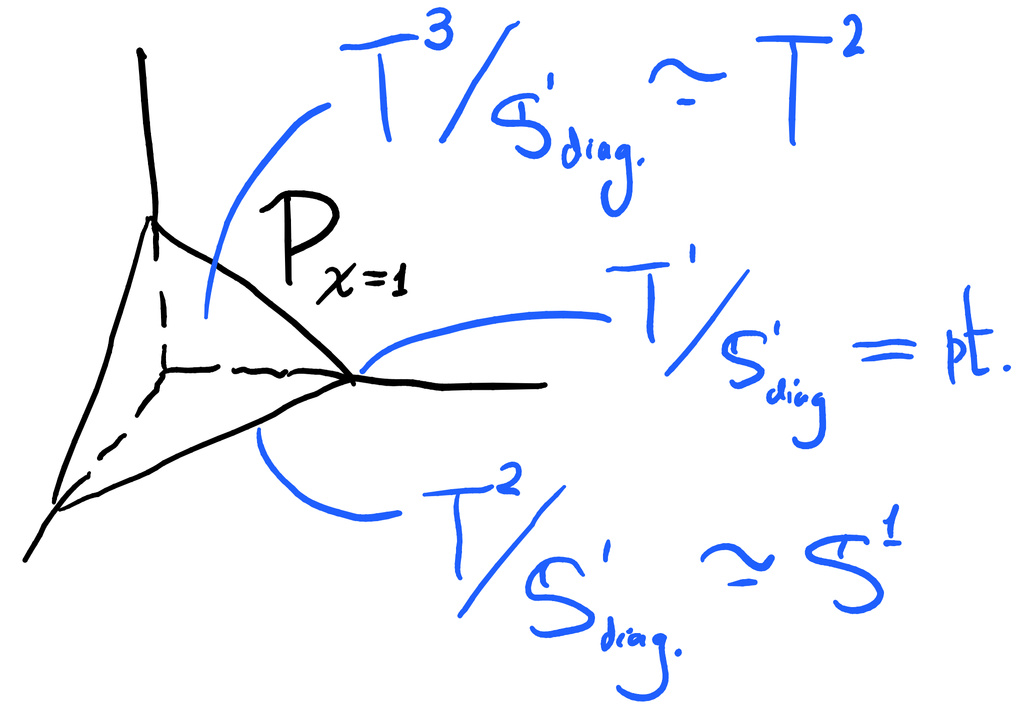
e.g.



$K \curvearrowright X \xrightarrow{\mu} \mathfrak{k}^*_{\mathbb{R}}$.

Moment map $\mu : \tilde{X} \longrightarrow \mathfrak{k}^*_{\mathbb{R}}$: unique up to transl. in $\mathfrak{k}^*_{\mathbb{R}}$.

- $d(\langle \mu, \xi \rangle) = \iota_{\rho(\xi)}\omega$ (Ham.)
- K-equiv. ($\Rightarrow$ K-inv. if Abelian $\Rightarrow$ $\mu^{-1}\{\chi\}$ preserved by K)



e.g. $T^n \curvearrowright \mathbb{C}^n \xrightarrow{|z_i|^2/2} \mathbb{R}^n_{\geq 0}$. $|z_i|^2$ are inv. under T.

$$q_i dq_i + p_i dp_i = \iota_{\xi_i}\Big(\overbrace{\textstyle\sum_j dp_j \wedge dq_j}^{\omega}\Big),$$
$$\xi_i = -p_i \partial_{q_i} + q_i \partial_{p_i} : \mathfrak{t} \curvearrowright \mathbb{C}^n.$$

$$\begin{tikzcd}
K \subset T^n \curvearrowright \mathbb{C}^n \arrow[r, "\mu"] \arrow[rr, bend left, "\bar\mu"] & \mathbb{R}^n_{\geq 0} \arrow[r] & \mathfrak{k}^*_{\mathbb{R}}
\end{tikzcd}.$$

$$d(\langle \bar\mu, \xi \rangle) = d(\langle \mu, \xi \rangle) = \iota_\xi \omega.$$
$\xi \in \mathfrak{k}_{\mathbb{R}} \subset \mathbb{R}^n$

K-inv. $\Rightarrow \eta_\# \mu^\xi = \omega(\xi_\#, \eta_\#) = 0$ $\forall \xi, \eta$.
Orbit is isotropic, hence at most dim. n.
Also orbit $\subset$ fiber of μ.
$\therefore$ If $\dim K = n$ and the action is free, μ is a Lag. fib.

Motivation (Hamiltonian dynamics)

$\cancel{X =} \mathbb{R}^3$ our space.

$X\ \cancel{M} = T^*\mathbb{R}^3 = \mathbb{R}^3 \times (\mathbb{R}^3)^* \ni \underbrace{(q}_{\text{position}}, \underbrace{p)}_{\text{momentum}}$ 'phase space' of a particle.

How can we talk about momentum without meaning the velocity? Momentum of a particle at q assigns to each $v \in T_q\mathbb{R}^3$ the 'impact along v' $\in \mathbb{R}$. Hence it is a covector.

measure the 'impact' (i.e. kinetic energy)

Newton's law: $\overbrace{F}^{\text{vector field}}(q(t)) = \overbrace{m}^{\text{constant}} q''(t)$. 2^{nd} order ODE.

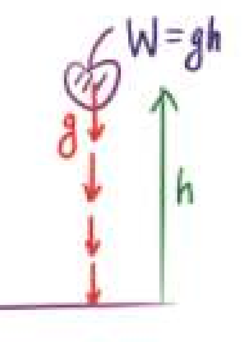
written as 1^{st} order ODE system: $\begin{cases} q'(t) = \frac{1}{m} p(t). \\ p'(t) = F(q(t)). \end{cases}$

Suppose the force field F have a 'source':

$$\sum_i F_i(q)\, dq_i = -d\underset{\text{called potential}}{W}.$$

e.g. $F\, dh = -g\, dh$.

$W = gh$.



Take $-H(q,p) = \frac{\|p\|^2}{2m} + W(q)$.

(Total energy, or Hamiltonian.)

$$-dH = \sum_i \frac{p_i}{m} dp_i - \sum_i F_i(q)\, dq_i \longleftarrow \text{closed 1-form}$$

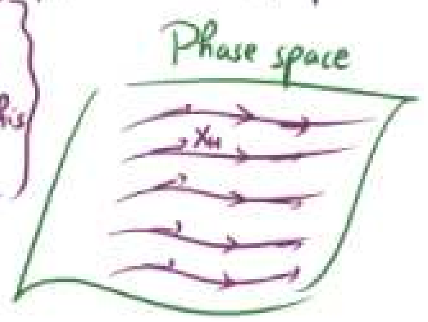
$$\xrightarrow[\ \omega\ ]{} \underbrace{X_H}_{\text{vector field}} = \sum_i \frac{p_i}{m} \frac{\partial}{\partial q_i} + \sum_i F_i(q) \frac{\partial}{\partial p_i} \longleftarrow \text{sympl. v.f.}$$

setting $\boxed{\omega = -\sum_i dq_i \wedge dp_i}$

$\therefore$ Newton's law

$\Leftrightarrow$ Hamiltonian flow $\binom{q}{p}'(t) = X_H\left(\binom{q}{p}(t)\right)$.

Want to formulate a global theory out of this structure.



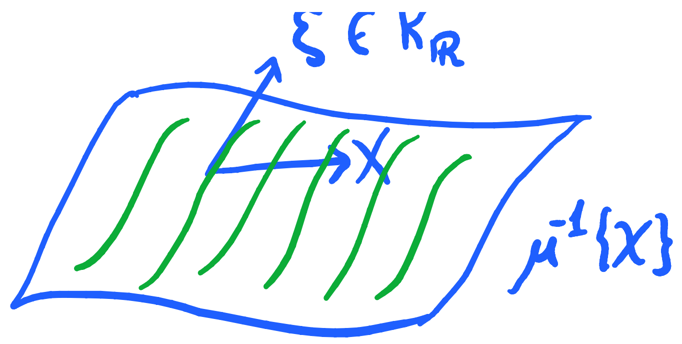
Note: H is conserved. $\iota_{X_H}\omega = dH$

$dH(X_H) = X_H H = \omega(X_H, X_H) = 0$.

$$\mathbb{R} \curvearrowright (M, \omega) \xrightarrow{H} \mathbb{R}.$$

$H^{-1}\{x\}/\mathbb{R}$: phases of same energy 'unrelated' to each other.

Sympl. str. ω descends to $\tilde{X}/\!\!/_x K$:



$$\bar{\omega}(\bar{X}, \bar{Y}) \triangleq \omega(X, Y).$$

$$\omega(X+\xi, Y+\eta) = \omega(X,Y) + \underbrace{\omega(\xi, Y)}_{\cancel{\partial_Y(\mu,\xi)}} + \underbrace{\omega(X,\eta)}_{\cancel{-\partial_X(\mu,\eta)}} + \underbrace{\omega(\xi,\eta)}_{\cancel{\partial_\eta(\mu,\xi)}}.$$

(const)

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symmetries of $(M, H) \longleftrightarrow$ conserved quantities

$$X_\varphi \cdot H = \pm\omega(X_H, X_\varphi) = X_H \cdot \varphi.$$

e.g. $X_\varphi = \frac{\partial}{\partial q_i} \longleftrightarrow -\varphi = p_i$. lin. momentum

$X_\varphi = -q_2 \partial_{q_1} + q_1 \partial_{q_2} \longleftrightarrow -\varphi = \det\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}$ angular momentum.

$-p_2 \partial_{p_1} + p_1 \partial_{p_2}$

Generalization:

symmetries of $(X, \mu) \longleftrightarrow$ G-inv. fns

$$X_\varphi \cdot \mu^X = \pm\omega(X_\varphi, X_\#) = X_\# \cdot \varphi.$$