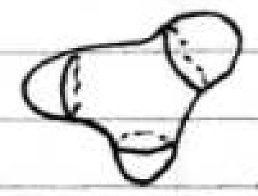
Wednesday, March 30, 2016 7:39 PM

§1. Overview.

Mirror symmetry: ([Candelas et. al.])

(X, ω)	$(\check{X}, \check{J})$
G.W. inv. $QH^{\cdot}$	Periods $H^{\cdot}_{\bar\partial}$
(counting of spheres) pseudo-hol.	(integration of hol. vol. form)

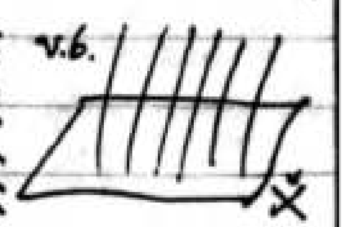


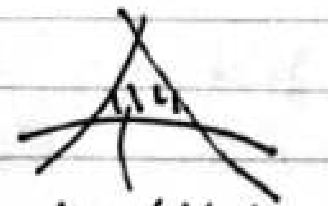
— originally for C.Y., discovered by string theorists.
later for many other types of varieties as well.

— open string version: $D\mathrm{Fuk}(X, \omega) \sim D\mathrm{Coh}(\check{X}, \check{J})$.

[Kontsevich. HMS]

objects: 



morphisms: 

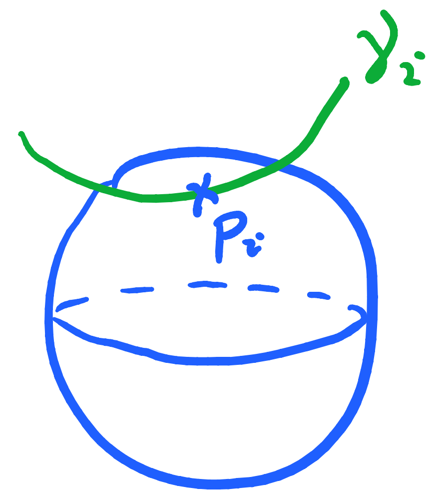
counting discs bdd by Lagrangians

$S_3 = S_1 \cdot S_2$
multi.$^{\text{ion}}$ of sections

1^{st} example: $X = \underbrace{\{z_0^5 + \dots + z_4^5 = 0\}}_{\text{quintic C.Y.}} \subset (\mathbb{P}^4, \varphi\omega_{FS})$

mirror quintic: $\check{X} = \{\underbrace{z_0^5 + \dots + z_4^5 + \psi z_0 \dots z_4}_{=0}\} / \mathbb{Z}_5^3$ $\quad \{(a_0 \dots a_4) \in \mathbb{Z}_5^5 : \Sigma a_i = 0\} / \{\text{diagonal}\}$

G.W. Choose J (almost cpx. str.) compatible with ω.



$$\mathcal{M}_{0,k}(\underset{\in\, H_2(X)}{\alpha}) = \{u : (\mathbb{P}^1, p_1 \dots p_k) \longrightarrow X : du \circ J_{\mathbb{P}^1} = J \circ du, \ [u] = \alpha\} / \mathrm{Aut}(\mathbb{P}^1).$$

$$\Big\downarrow ev_i(u) = u(p_i) \quad i = 1 \dots k$$

$$X$$

$$\langle \gamma_1 \dots \gamma_k \rangle \triangleq \sum_{\alpha} T^{\omega(\alpha)} (ev_1 \times \dots \times ev_k)_* [\mathcal{M}_{0,k}(\alpha)] \cdot (\gamma_1 \cup \dots \cup \gamma_k).$$

$$(T \sim e^{-1})$$

$$(\gamma_1 * \gamma_2, \gamma_3) \triangleq \langle \gamma_1, \gamma_2, \gamma_3 \rangle.$$

$QH^{\cdot}(X) = (H^{\cdot}(X), *, (\cdot,\cdot))$ Frobenius algebra.

$T = 0$: only $\alpha = 0$ contribute $\Longrightarrow$ $* = \cup$.

Main issue of $\mathcal{M}_{0,k}(\alpha)$:

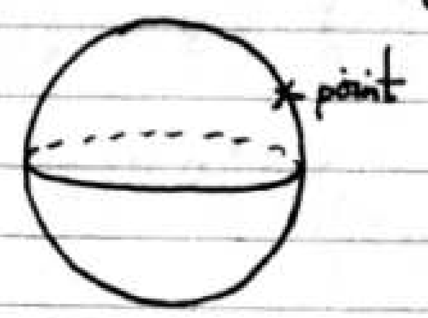
(1) compactification by stable holomorphic curves.

Need to include these in compactified moduli.

(2) transversality. $\exists$ obstruction bundle. Need virtual fundamental class.

An example to show the magic of M.S.

eg. S^2.



$$H^*(S^2) = \mathbb{Z}\langle \underbrace{[pt]}_{\text{point class}}, \underbrace{1}_{\text{fundamental class}} \rangle.$$

Classical ring str.: $1 \cup 1 = 1$, $[pt] \cup 1 = [pt]$, $[pt] \cup [pt] = 0$

(intersection btw. cycles)

(counting $\overset{?}{\text{constant}}$ spheres) $\quad$ $a \cup b$ '=' union of intersection points btw a and b.

Quantum ring str.: $\displaystyle a * b \triangleq \sum_{\alpha \in H_2} e^{-\int_\alpha \omega} \cdot \langle a, b, c_i^\vee \rangle_{0,3,\alpha} \cdot c_i$.

(counting spheres: genus 0 Riem. surf.) $\quad$ (# hol. spheres passing thru $a, b, c_i^\vee$)

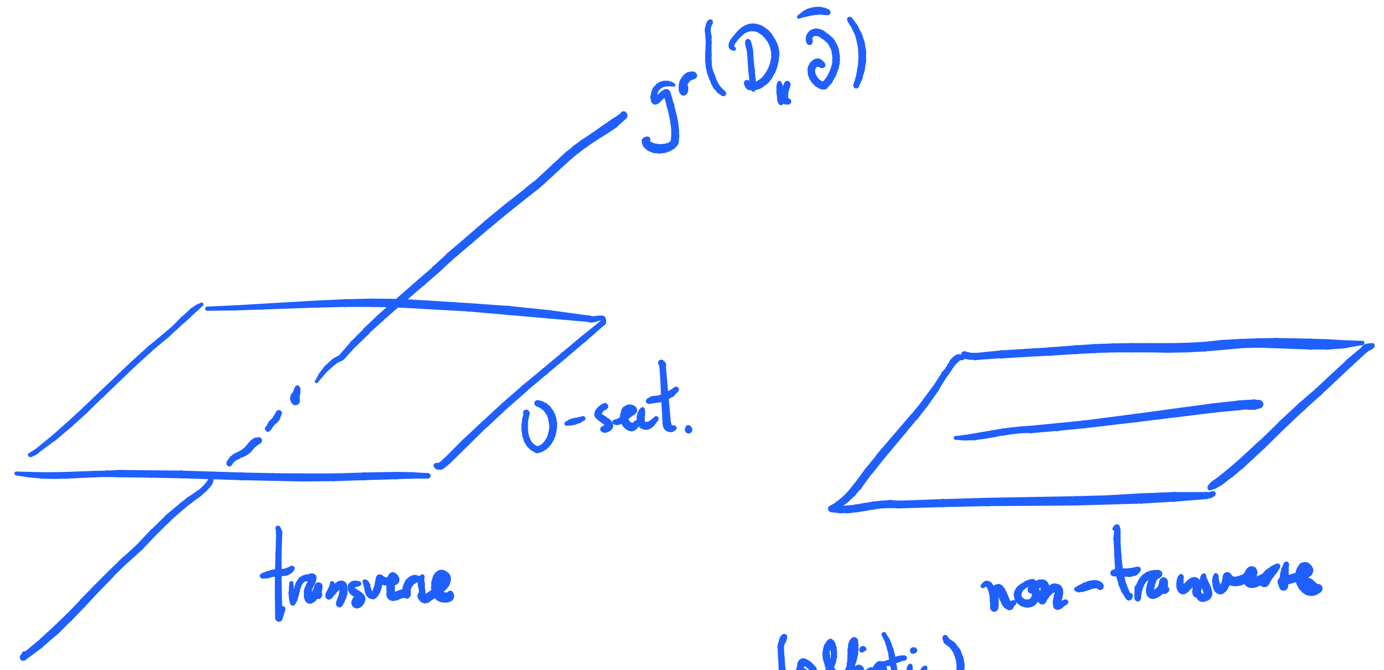
(denote the ring by $QH^*(S^2)$) $\quad$ $\{c_i\}$: basis of H^*

$\{c_i^\vee\}$: dual basis wrt Poincaré pairing.

$$1 * 1 = 1, \ [pt] * 1 = [pt], \ \underbrace{[pt] * [pt] = e^{-\int_{S^2} \omega}}_{\text{different from classical}}.$$

(due to $\underbrace{\text{dim. reason}}_{\text{explain later}}$, only spheres w/ deg 0 & 1 contribute.)

$$\Omega^{0,1}\big((\mathbb{P}^1, \{p_1 \ldots p_k\}), u^*(TX)\big)$$
$$\downarrow \quad \bigg) \bar\partial$$
$$\mathrm{Map}\big((\mathbb{P}^1, \{p_1 \ldots p_k\}) \xrightarrow[{[\alpha]}]{} X\big) \ni u$$



$\mathcal{M} = \mathrm{graph}(\bar\partial) \cap (0\text{-section})$.

$\uparrow$ may not be transverse

Obstr. bundle: $\mathrm{Coker}(D_u \bar\partial)$ at u.

(elliptic)

index $=$ ker $-$ coker (finite, assume ≥ 0)

(expected dim) $=$ ker if coker $= 0$. $\leftarrow$ transverse situation

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deg 0 : usual $\cup$ product

1 : $\langle p.p.p \rangle = T^{\omega(S^2)}$. $\quad (\dim \mathcal{M}_{0,3}(\deg = 1) = 3.)$

≥ 2 : 0 since $\underbrace{\dim_{\mathbb{C}} \mathcal{M}_{0,3}(\deg)}_{\underbrace{c_1}_{2\cdot\deg} + 3 + 1 - \underbrace{3}_{\dim \mathrm{Aut}(\mathbb{P}^1) \,=\, PGL(2,\mathbb{C})}} > 3$.

Mirror to S^2:

$$\left(\mathbb{C}^\times,\ W = z + \frac{\check{q}}{z}\right). \quad \text{(L.G. model)}$$

$$\mathrm{Jac}(W) = \mathbb{C}[z, z^{-1}] \Big/ \left\langle z - \frac{\check{q}}{z} \right\rangle.$$

Ring str.: Generators 1 and $z \sim \frac{\check{q}}{z}$.

$$1 \cdot 1 = 1, \quad z \cdot 1 = z, \quad z^2 \sim \check{q}.$$

Same as that of $QH^*(S^2)$!

The dimension formula that I have used:

$$\underset{\text{(expected)}}{\dim} \mathcal{M}_{0,k,\alpha} = c_1(\alpha) + \dim_{\mathbb{C}} X + k - 3.$$

genus, marked pts in our example $= 2 \cdot \deg \alpha + 1.$

$(c_1(S^2) = 2[pt])$

$(k=3)$

Require $2 \cdot \deg\alpha + 1 - \operatorname{codim} a - \operatorname{codim} b - \operatorname{codim} c = 0.$

at most 1

$\therefore \deg \alpha \leq 1.$

$\deg \alpha = 1 \Rightarrow a, b, c$ are pts.

(quantum contribution)

($\deg\alpha = 0$ is classical part)

[Hori-Vafa]

Landau Gizburg mirror for general compact toric mfd $X = \mathbb{C}^m /\!\!/_{\chi} K$:

$$0 \to H_2(X) \to H_2(X, T) \to N \to 0.$$

$$\left(\operatorname{Spec} \mathbb{C}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \big/ \sim,\ W = \sum_{i=1}^m z_i\right)$$

where $\forall \sum_{i=1}^m a_i v_i = 0$, $a_i \in \mathbb{Z}$.

$$\prod_{i=1}^m z_i^{a_i} \sim q^\alpha, \quad \alpha = \sum_{i=1}^m a_i \beta_i \in H_2(X).$$

(i.e. $(\alpha, D_i) = a_i$.)

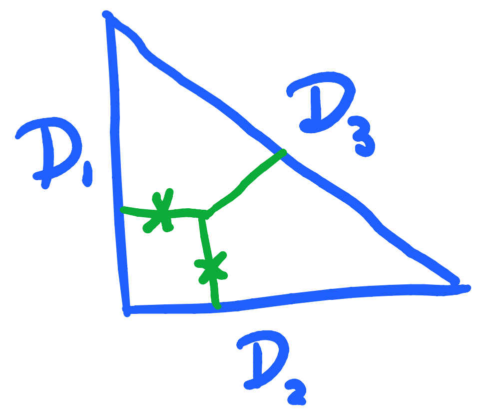
can take $z_1^\pm \ldots z_n^\pm$ corr. to a basis $v_1 \ldots v_n$ and express all other monomials in $z_1^\pm \ldots z_n^\pm$.

Mirror thm.: $\underset{\uparrow \text{Fano}}{QH(X)} = \mathrm{Jac}(W)$. [Givental; Batyrev]

(set $q^\alpha = T^{\omega(\alpha)}$.) $\quad \|$ (generalized to non-Fano toric by [FOOO].)

$$\mathbb{C}[z_1^\pm \dots z_n^\pm] \big/ \langle \partial_i W, \ i=1\dots n\rangle.$$

e.g. $QH(\mathbb{P}^2) \simeq \mathbb{C}[z_1^\pm, z_2^\pm] \big/ \langle z_1 \sim \frac{q}{z_1 z_2} \sim z_2 \rangle$



$D_1 \sim D_2 \sim D_3 \longmapsto z_1 \sim z_2 \sim \frac{q}{z_1 z_2}$.

$\underbrace{D_1 * D_2 * D_3}_{P} = q = T^{\omega(\ell)}$. (Only non-trivial 3-pt GW: $\langle p, p, D_i \rangle = 1$.)

$$\dim_{\mathbb{C}} \mathcal{M}_{0,3}(d) = \overset{c_1}{\overset{\downarrow}{3d}} + \overset{\#\mathrm{pt}}{\overset{\downarrow}{3}} + \overset{\dim X}{\overset{\downarrow}{2}} - 3$$

d	$\dim_{\mathbb{C}} \mathcal{M}$	
0	2	$(p,1)$ or (D,D)
1	5	(p,p,D_i)
2	8	(too big)

ex. Find $QH(\mathbb{F}_1)$.