§ 1. Overview.

Mirror symmetry: ([Candelas et al.])

\[
\begin{align*}
\text{GW. } & \text{Choose } J (\text{almost cpx. str.}) \text{ compatible with } \omega. \\
\mathcal{M}_{0,k}(\alpha) &= \{ u : (\mathbb{P}^1, p_1, \ldots, p_k) \to X : du \circ J = J \circ du, \\
&\quad [u] = \alpha \} \\
&\quad \text{Aut}(\mathbb{P}^1).
\end{align*}
\]

\[
\begin{align*}
\mathcal{H}^0(\mathcal{V})^k &\to X : du \circ J = J \circ du, \\
\text{ev}_i(u) &= u(p_i) \quad i = 1, \ldots, k
\end{align*}
\]

\[
\langle \gamma_1, \ldots, \gamma_k \rangle \equiv \prod \omega(\alpha)_i [\mathcal{M}_{0,k}(\alpha)] \cdot (\gamma_1, \ldots, \gamma_k).
\]

\[
(\gamma_1, \gamma_2, \gamma_3) \equiv \langle \gamma_1, \gamma_2, \gamma_3 \rangle.
\]

\[
\mathcal{QH}^*(X) = (\mathcal{H}^*(X), \ast, \cdot), \text{ Frobenius algebra.}
\]

\[
\prod = 0 : \text{ only } \alpha = 0 \text{ contribute } \Rightarrow \ast = 0.
\]
Main issue of $M_{0,k}(X)$:

1. Compactification by stable holomorphic curves.

An example to show the magic of $M_3$.

$\Omega^2((\mathbb{P}'(p_1\ldots p_k))_u(TX))\overset{\overline{\partial}}{\longrightarrow}\text{graph}(\overline{\partial}) \cap (0 \text{-section})$

Transverse may not be transverse for $\overline{\partial}$.

Obstr. bundle: $\text{Coker}(D_u\overline{\partial})$ at $u$.

$\text{deg } 0$: usual $u$ product

$1: \langle p, p, p \rangle = \prod_{i<j<k} \omega_i(\delta, \delta)$ (dim $M_{0,3}(\text{deg}=1)=3$)

$>2: 0$ since $\dim_c M_{0,3}(\text{deg}) > 3$.

\[
\frac{c_2 + 3 + 1 - 3}{2 \cdot \text{deg}} \geq \dim \text{Aut}(\mathbb{P}^1)
\]

PGL(2, C)
$\text{Mirror to } S^2$: 
\[ (C^*, W = z + \frac{x}{2}) \quad (\text{L.H. model}) \]

\[ \text{Jac}(W) = C[z, z^2] / \langle z - \frac{x}{2} \rangle. \]

Ring str.: Generators $1$ and $z^\frac{x}{2}$.

$1 \cdot 1 = 1, \quad z \cdot 1 = z, \quad z^2 \sim 1.$

Same as that of $\text{OPH}(5)!$

The dimension formula that I have used:

\[
\dim M_{0,k,x} = \dim C^* + \dim C^* + 3 - 3.
\]

Example:

\[
\dim \text{cone}(C^* / C^* = 2 \cdot \text{deg} + 1.
\]

$\text{deg} = 2 \implies \dim = 2 \cdot \text{deg} + 1.$

\[
(k = 3)
\]

$\text{deg} = 1 \implies 2 \cdot \text{deg} + 1 = 0.$

\[
\text{deg} = 0 \implies \text{deg} = 0.
\]

\[
\text{deg} = 1 \text{ at most } 1.
\]

\[
\text{deg} = 1 \implies a, b, c \text{ are pts.}
\]

(Quantum contribution)

\[
(\text{deg} = 0 \text{ is trivial part})
\]

\[ \text{[Hori-Vafa]} \]

\[ \text{Landau-Ginzburg mirror for general compact} \]

\[ \text{toric ufd } \quad X = C^* / K : \]

\[
0 \to H_2(X) \to H_2(X, T) \to N \to 0.
\]

\[ (\text{Spec } C[z_1, \ldots, z_m] / \nu, \quad W = \sum_{i=1}^m z_i) \]

\[ \text{where } \quad \sum_{i=1}^m a_i v_i = 0. \]

\[ \prod_{i=1}^m 2_i \sim \nu \cdot a = \sum_{i=1}^m a_i \nu_i \in H_2(X). \]

(i.e. $(a, \nu_i) = a_i$.)

\[ \text{Can take } 2^i, \ldots, 2^m \]

\[ \text{correspond to a basis } v_1, \ldots, v_m \]

and express all other monomials in $2^n, \ldots, 2^m$. 

\[ \text{in } z_1^\pm \ldots z_m^\pm. \]
Minor hint: $QH(X) = \text{Jac}(W) \uparrow_{\text{Fano}} \quad \text{[Givental; Batyrev]}
\quad \left( \text{set } q^* = \prod_{w} \right)
\quad \text{all (generalized to non-Fano case by [F00])}
\quad \mathbb{C}[z_1^* \ldots z_n^*] \quad \langle \partial_i W, i = 1 \ldots n \rangle.$

\text{e.g. } QH\left(\mathbb{P}^2\right) \sim \mathbb{C}[z_1^*, z_2^*] \quad \langle z_1 \sim \frac{q}{z_1 z_2} \sim z_2 \rangle

\text{D}_1 \sim \text{D}_2 \sim \text{D}_3 \quad \rightarrow \quad z_1, z_2 \sim \frac{q}{z_1 z_2} \quad \text{D}_1 \ast \text{D}_2 \ast \text{D}_3 = q = \prod_{w}^{\text{orb}} \quad \text{(Only non-trivial 3-pt GW: } \langle p, p, \text{D}_i \rangle = 1 \text{.)}

\text{ex. Find } QH\left(\mathbb{P}_1\right).