Friday, February 5, 2016 11:52 AM

S.Y.Z. fantasy.

Mirror symmetry is T-duality.

v.s.:	V	V^*
compactify:	V/Λ	$V^*/\Lambda^* = \{\text{flat } u(1) \text{ conn. on } V/\Lambda\}$
family version:	TB/Λ_B	$T^*B/\Lambda_B^* = \{\text{fiberwise flat } u(1) \text{ conn.}\}$. ← Toy model of M.S.

B : tropical affine mfd.

change of coord. $\in GL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$.

$\Lambda_B \doteq \mathbb{Z}\langle \frac{\partial}{\partial r_1}, \dots, \frac{\partial}{\partial r_n} \rangle$. ($r_i$'s are local affine coord.)

$\Lambda_B^* \doteq \mathbb{Z}\langle dr_1, \dots, dr_n \rangle$.

KEY DIFFERENCE: singular fibers.

Key : TB has canonical complex str. and

T^*B has canonical symplectic str. !

[Leung, Leung-Yau-Zaslow] semi-flat M.S. ← $\boxed{\text{talk later}}$

Strominger-Yau-Zaslow conjecture:

For every mirror pair of C.Y. manifolds $(X, \check{X})$,

X and $\check{X}$ have special Lagrangian fibrations over the same base B

which are dual to each other.

e.g. torus.

Why important :

All Lag. fib. is like T^*B/Λ_B^* !

away from sing. fibers

[Arnold-Liouville action-angle coordinates]

S. Lag. fibrations.

(X, ω, J, g) Kähler. for gen. Käh., can consider $X - D$. $D \in |-K_X|$. [Auroux07] don't require $\omega^n \sim \Omega \wedge \bar{\Omega}$

C.Y. means $c_1(TX) = 0$. Pick hol. vol. form $\Omega \in H^0(K_X)$.

$L \subset X$ is Lag. if $\omega|_L = 0$.

" S.Lag. (with phase θ) if $\mathrm{Im}(e^{i\theta}\Omega)|_L = 0$ for some $\theta \in \mathbb{R}$.

e.g. Any $\mathbb{R}$-curve in $(\mathbb{C}, \omega_{std}, J_{std}, g_{std}, \underset{dz}{\Omega_{std}})$ is Lag.,

$\exists$ Ham. deform. while $\underbrace{\text{straight lines}}_{\text{'can.' rep. in the class.}}$ are S.Lag.

$X \downarrow B$ is a S.Lag fib. if vertical tangent spaces are special Lag.

[Arnold - Liouville] A proper Lag. fib. w/ conn. fibres is torus fib. B has aff. str.

$$\begin{tikzcd} X \arrow[dr, "\pi"'] & & \check{X} \arrow[dl, "\check{\pi}", "\text{S.Lag.}"'] \\ & B & \end{tikzcd}$$

are dual if they have the same disc. locus, and

$$\begin{tikzcd} \check{\pi}^{-1}(B_0) \arrow[r, "\simeq"] \arrow[d, "\check{\pi}"] & \{\text{flat } \mathrm{U}(1) \text{ conn. on } \pi^{-1}(B_0)\} \arrow[dl] \\ B_0 & \end{tikzcd}$$

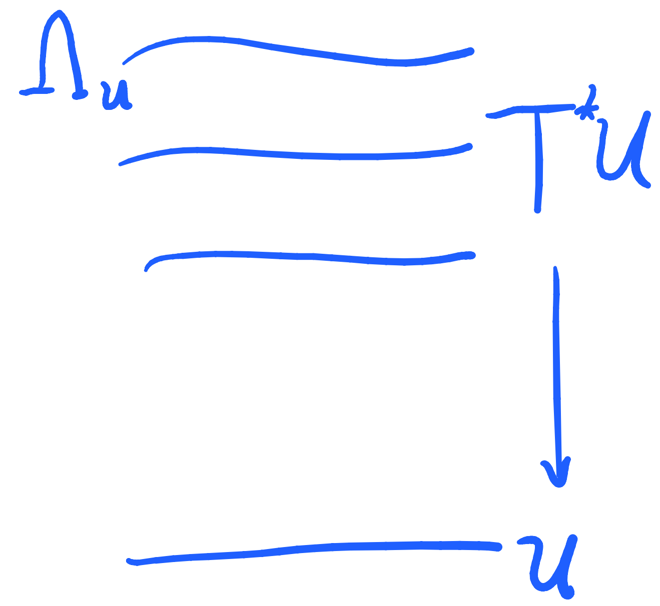
$B_0 = B -$ disc. locus

More on S.Lag. [McLean] [Hitchin] [Gross] Ch. 6.

Talk about it more

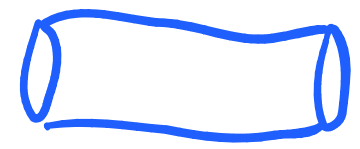
[Arnold - Liouville]

$$\begin{tikzcd} X \arrow[d, "\pi : \text{ proper Lag. submer.}"] \\ B \end{tikzcd}$$



Then $\forall p \in B$, $\exists\, U \ni p$ st.

$$\begin{tikzcd} T^*U/\Lambda_U \arrow[rr, "\sim"] \arrow[dr] & & \pi^{-1}(U) \arrow[dl] \\ & U & \end{tikzcd}$$



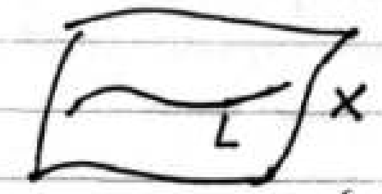
If π has a global section, then U can be taken to be B.

Key: Any $\eta \in T_b^*U$ gives $(\pi^*\eta)_{\#_\omega}$: v.f. along $\pi^{-1}\{b\}$.

$$\left(\omega\left((\pi^*\eta)_{\#\omega}, Y\right) = (\pi^*\eta)(Y) = \eta(\pi_* Y) = 0 \quad \forall Y \text{ along } \pi^{-1}\{b\}.\right.$$

§2. Deformation of s.Lag. & affine geometry.

$L \underset{\text{Lag., cpt. conn.}}{\subset} (X, \omega)$.



ref: §8.1 of [Dirichlet branes and M.S.]. [McLean] [Hitchin]

$\omega|_L = 0$.

Weinstein tubular neighborhood thm: $\left(\begin{array}{l}\text{Moser trick} \\ + \text{lin. alg.}\end{array}\right)$

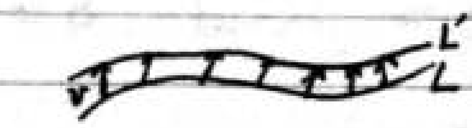
$$\text{nbd of } L \underset{\text{Lag.}}{\subset} X \underset{\text{sympl}}{\simeq} \text{nbd of } L \underset{\text{Lag.}}{\subset} T^*L \underset{\omega}{\cong} NL$$

$\therefore$ (small) deformation of L can be described by 1-forms $\Omega^1(L)$.

Require the deformation is still Lag.:

Denote by v the v.f. corr. to the inf. deform.

$$\mathcal{L}_v \omega = 0.$$
$$\| \ (d\omega = 0)$$
$$d(\iota_v \omega)$$



$\therefore$ Lag. deform. of L described by $\Omega^1_{closed}(L)$.

Identify the Hamiltonian deform.:

Ham. v.f. corr. to df on X.

Restricting to L, get $d(f|_L)$.

$\therefore$ Ham. deform. of L described by $\Omega^1_{exact}(L)$.

$\therefore$ nbd of $\dfrac{\text{Lag deform. of } L}{\text{Ham. deform. of } L} \simeq$ nbd of zero in $H^1(L)$.

$$\boxed{\text{Ham. class of } L \longleftrightarrow \text{Small element in } H^1(L)}$$

- Want canonical representative instead of class!

  (subspace is easier to understand than quotient.)

- Use metric.

  hol. volume form

  Equip X with (g, J, Ω). $\quad (\Rightarrow c_1(TX) = 0$ C.Y. cond.)

  $\Omega \in \Gamma(K_X)$

  $((X, g, J, \omega)$ Kähler.) $\quad \left(\text{Assume } \dfrac{\omega^n}{n!} = \Omega \wedge \overline{\Omega}\right)$

  Let L be s.Lag. : $\omega|_L = 0$, $\operatorname{Im}\Omega|_L = 0$.

  s.Lag. deform. of L : in addition $\mathcal{L}_v(\operatorname{Im}\Omega) = 0$.

  $$\mathcal{L}_v(\operatorname{Im}\Omega) = d(\iota_v \operatorname{Im}\Omega)$$

Important identity : $\boxed{\iota_v \operatorname{Im}\Omega = -(*_L \iota_v \omega)}$.

Then $\iota_v\omega \in \Omega^1(L)$ is harmonic (w.r.t. $g|_L$).

$\therefore$ nbd of $\{$s.Lag. deform. of $L\} \simeq$ nbd of $\mathcal{H}^1(L)$.

Deriving $* \iota_v \omega = -\iota_v \operatorname{Im}\Omega$. $\quad \sum_i (dx_i \wedge dy_i)$

Consider $X = \left(\mathbb{C}^n,\ \omega = \frac{i}{2} \sum_{k=1}^n dz_k \wedge d\bar{z}_k,\ \Omega = dz_1 \wedge \dots \wedge dz_n\right)$.

$L = \mathbb{R}^n \subset \mathbb{C}^n$. $\quad g_{std}$

$$v = \sum a_i \partial_{y_i} . \qquad (z_i = x_i + i y_i).$$

$$(\iota_v \operatorname{Im}\Omega)|_L = \sum_{i=1}^n (-1)^{i+1} \operatorname{Im} dz_1 \wedge \dots \wedge (i a_i) \wedge \dots \wedge dz_n \Big|_L$$

$$= \sum_{i=1}^n (-1)^{i+1} a_i \, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n .$$

$$= \sum_{i=1}^n a_i *_L (dx_i) .$$

$$(\iota_v \omega)|_L = -\sum_{i=1}^n a_i \, dx_i .$$

$$\therefore * \iota_v \omega = -\iota_v \operatorname{Im}\Omega$$

In general $(T_pM, \omega, \Omega, g) \simeq (\mathbb{C}^n, \omega_{std}, \Omega_{std}, g_{std})$.

$$\begin{array}{lccc} & & SU(n) & \\ \omega|_L = 0 & \cup & & \cup \\ \operatorname{Im}\Omega|_L = 0 & L & \longrightarrow & \mathbb{R}^n \end{array} \qquad \#$$

(argue by choosing basis) $\quad$ use the assumption $\frac{\omega^n}{n!} = \Omega \wedge \bar{\Omega}$.

otherwise $* \iota_v \omega = -\beta \iota_v \operatorname{Im}\Omega$.

$\iota_v \omega$ is harm. wrt ι-df.

McLean's thm.: $\{$s. Lag. deform. of $M\}$ is a mfd,

(Thm. 6.2) $\quad$ whose tangent sp $\simeq \mathcal{H}^1(M, \mathbb{R})$.

harmonic

Induced strs on the base.

Let B : family of s.Lag. mflds.

Have $T_{[L]}B \longrightarrow \mathcal{H}^1(L_y, \mathbb{R})$.

Assume this is $\simeq$.

Induced str. on B :

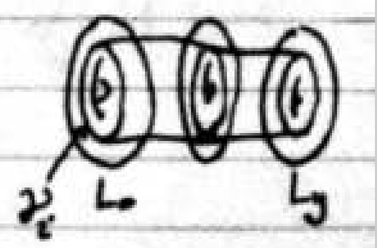
① Affine coord. (symplectic) :

Fix $\underset{y_0}{[L_0]} \in B$, and a basis $[\gamma_i]$ of $H_1(L_0, \mathbb{Z})/_{\text{tor}}$.

Take $\underset{B}{\overset{\text{open}}{\cap}} \mathcal{U} \ni [L_0]$ contractible s.t. s.Lag. para. by $\mathcal{U}$ are contained in a tubular nbd of L_0.

Then $[\gamma_i]$ can be translated as classes $\in H_1(L_y, \mathbb{Z})/_{\text{tor}}$ $\forall y \in \mathcal{U}$.

$$r_i(y) \doteq \int_{\underset{\text{path joining } y \text{ and } y_0}{\overset{\uparrow}{\cup_t \gamma_i(t)}}} \omega .$$



well-def. because $d\omega = 0$, and L's are Lag.

from a basis of Harm 1-forms

$$\frac{\partial r_i}{\partial y_j} = \int_{\gamma_i} \iota_{\frac{\partial}{\partial y_j}} \omega .$$

$\therefore$ Jac is non-deg

$\Rightarrow r_i$ are loc. coord.

Chose other $[L_0'] \in B$

$\rightsquigarrow r_i'(y) = r_i(y) + \text{const}^{\in \mathbb{R}}$.

Choose another basis $[\gamma_i']$

$$\longrightarrow \; r_i' = \sum_j \overset{\in GL(n,\mathbb{Z})}{A_{ji}} r_j$$

$\therefore$ Change of coord. is trop. aff. $(GL(n,\mathbb{Z}) \ltimes \mathbb{R}^n)$.

Note : We don't need 'special' to obtain this str.

(2) Affine coord. (cpx) :

Take basis $\{\Gamma_i\}$ of $H_{n-1}(L_{y_0}, \mathbb{Z}) / \mathrm{tor}$.

$$R_i(y) \triangleq \int_{\underset{\text{path joining } y \,\&\, y_0}{\cup \Gamma_i(t)}} (\mathrm{Im}\,\Omega).$$

$$\partial_{y_j} R_i = \int_{\Gamma_i} \underbrace{\iota_{\frac{\partial}{\partial y_j}} \mathrm{Im}\,\Omega}_{-* \iota_{\frac{\partial}{\partial y_j}} \omega \text{ form a basis of } \mathcal{H}^{n-1}(L_0)}.$$

$\therefore$ R_i defines trop. aff. coord.

B has two affine str. !

③ McLean metric.

Pull back the metric by

$$T_{[L]}B \xrightarrow{\sim} \mathcal{H}^1(L, \mathbb{R}).$$

Conclusion: Lag. deform $\longrightarrow$ aff. str. on def. sp.

s. Lag. deform. $\longrightarrow$ two aff. strs, or metric on the deform. sp.

Relation btw/ two affine str.: Leg. Transform.

r: sympl. aff. coord.

$\exists$ loc. fcn. $g(r)$ s.t. ~~R~~

$$\begin{cases} R_i = g_{,r_i}, \text{ and} \\ \underbrace{\left\langle \frac{\partial}{\partial r_i}, \frac{\partial}{\partial r_j} \right\rangle}_{\text{McLean metric}} = g_{,r_i r_j} \end{cases}$$

locally exist since they are coord.

Moreover, let $G(R) \triangleq \sum_i R_i \, r_i(R) - g(r(R))$

be the Legendre transform of g.

Then $$\begin{cases} r_i = G_{,R_i} \\ G_{,R_i R_j} = \left\langle \frac{\partial}{\partial R_i}, \frac{\partial}{\partial R_j} \right\rangle. \end{cases}$$

Pf: Take $\{\gamma_i\}, \{T_i\}$ to be P.D.:

$$\gamma_i \cdot T_j = \delta_{ij}.$$

$$\gamma_i \to r_i, \quad T_j \to R_j$$

$r_i = \int_{\gamma_i} \omega.$

$[\iota_{\partial r_i} \omega]$ P.D. to T_i:

$\int_{\gamma_j} \iota_{r_i} \omega = dr_j\left(\frac{\partial}{\partial r_i}\right) = \delta_{ij}.$

$$\langle \partial_{r_i}, \partial_{r_j} \rangle = \int_{F_r} (\iota_{\partial r_i} \omega) \wedge *(\iota_{\partial r_j} \omega)$$

$$= -\int_{F_r} (\iota_{\partial r_i} \omega) \wedge (\iota_{\partial r_j} \operatorname{Im}\Omega).$$

$$= -\int_{T_i} \iota_{\partial r_j} \operatorname{Im}\Omega = R_{i, r_j}.$$

Then $\sum_i R_i \, dr_i$ is closed.

$\therefore \exists$ local g s.t. $dg = \sum R_i \, dr_i$
$$\underset{g_{,r_i} dr_i.}{\overset{\|}{dg}}$$

$$g_{,r_i r_j} = R_{i,j} = \langle \partial_{r_i}, \partial_{r_j} \rangle.$$

For $G(R) \doteq \sum_i R_i \, r_i(R) - g(r(R))$,

$$G_{,R_j} = r_j + \sum_i R_i \, r_{i,R_j} - \sum_k \underbrace{g_{,r_k}}_{R_k} r_{k,R_j}$$
$$= r_j.$$

$$g(\partial_{R_i}, \partial_{R_j}) = g\left(\frac{\partial r_\ell}{\partial R_i} \partial_{r_\ell}, \frac{\partial r_p}{\partial R_j} \partial_{r_p}\right)$$
$$= \frac{\partial r_\ell}{\partial R_i} \frac{\partial r_p}{\partial R_j} \underbrace{g_{,r_\ell r_p}}_{R_{\ell,r_p}} = \frac{\partial r_j}{\partial R_i} = G_{,R_j R_i}. \#$$