Friday, February 5, 2016 11:52 AM

§6. Lag. Floer theory on cpt toric mflds.

[ref.: Fooo
Cho-Oh
Auroux
Cho]

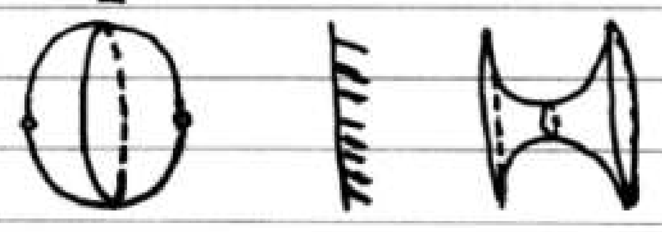
Motivation.

We have seen $T^*B/\Lambda^* \equiv TB/\Lambda$.

The closest geometry to this is X_Δ: toric mfd.

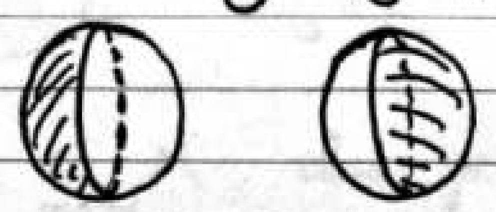
Ignoring sing. fibers $(T^*\Delta/N, \omega) \equiv (\mathbb{C}^\times)^n$.

$$\begin{array}{c} (T^*\Delta/N, \omega) \\ \cap \\ X_\Delta \end{array}$$



- To capture the missing singular fibers, consider

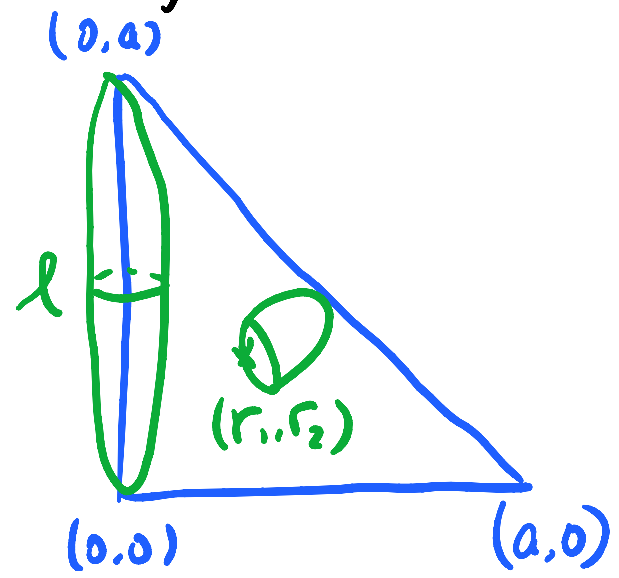
  # discs bdd by Lag. fibers (weighted by area).



This is Lag. Floer theory applied on fibers.
(Cho-Oh Fano.
Fooo general.)

$$T \curvearrowright X \xrightarrow[\mu]{} P \subset \mathfrak{t}^*.$$

e.g.



area of line $\ell = a$.

area of $\beta(u,v) = a - r_1 - r_2$.

linear coord. $X \in \mathfrak{t}_{\mathbb{Z}}$ is action coord. on P°:

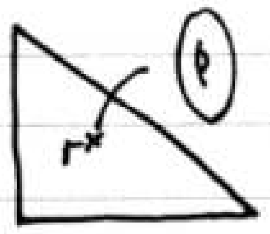
$$\int_{\partial_X} \iota_v \omega = \int_0^1 \omega_{\gamma_X(t)}(v_{\gamma_X(t)}, X \cdot \gamma_X(t))\, dt$$

$(v \in T_b P)$

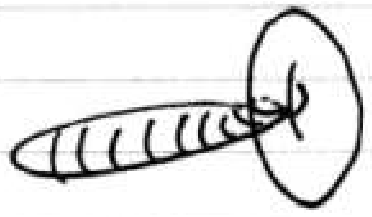
$$= -\int_0^1 \underbrace{v_{\gamma_X(t)} \cdot \underbrace{\mu^X}_{\mu^* X}}_{v \cdot X} dt = -v \cdot X.$$

Discs bdd by Lag. fibers.

$$\begin{array}{ccc} (X_\Delta, \omega) & \supset & F_r \\ \downarrow & & \downarrow \\ \Delta & \ni & r \end{array}$$



Consider $H_2(X, F_r)$.

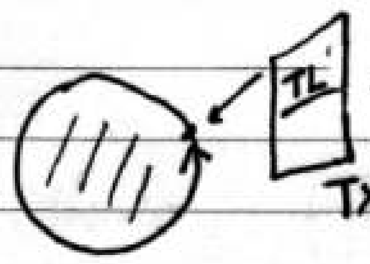


Two important classical symplectic inv. ass. to $\beta \in H_2(X, F_r)$

- sympl. area. $\int_\beta \omega$.

- Maslov index μ: $\left[\partial\beta \to LGr \xrightarrow{\det} U(1)/\mathbb{Z}_2\right] \in \pi_1(U(1)/\mathbb{Z}_2) \cong \mathbb{Z}$.

$LGr \cong U(n)/O(n)$



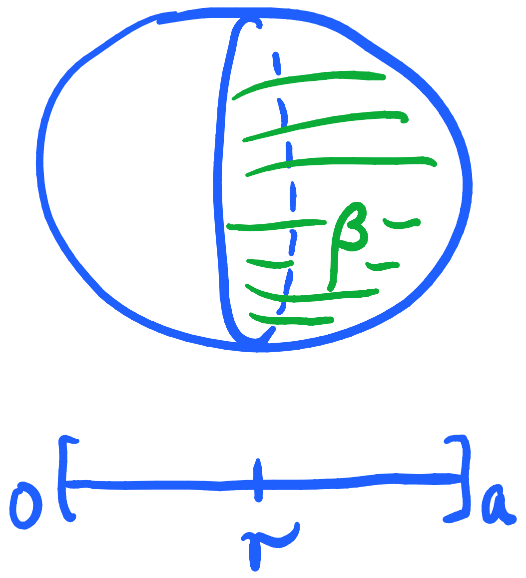
$TX|_\beta \simeq \mathbb{R}^{2n}$

- If $L \underset{\text{oriented}}{\subset} X$, then $\mu(\beta) \in 2\mathbb{Z}$.

μ is important for computing dim. of moduli of p.h. discs.

It is an analog of $c_1(\alpha)$ for $\alpha \in H_2(X)$.

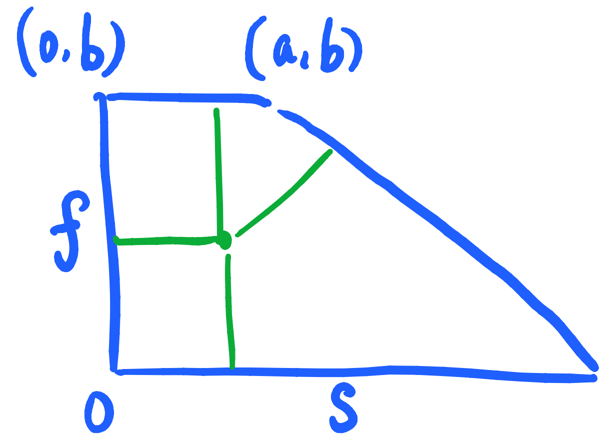
e.g. $\mathbb{P}^1$.



area of $\mathbb{P}^1 = a$.

area of $\beta = a - r$.

ex. $\mathbb{F}_1$.



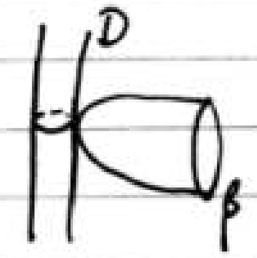
- What is the area of f and $s \in H_2(X)$?
- Compute the disc potential.
- Find a non-displaceble Lag. fiber.
- What are the areas of the basic discs bdd by the non-displaceble Lag. fiber?

Formula of Maslov index.

Lemma [Auroux] (Cho-Oh): Y Käh., dim. n.

Ω nowhere-zero mero. n-form.

D: pole div. of Ω.



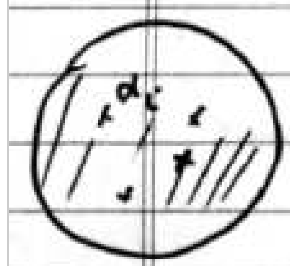
$L \subset Y - D$ cpt oriented s.Lag.

$\forall \beta \in \pi_2(Y, L)$, $\mu(\beta) = 2\beta \cdot D$.

Pf: $f: \Delta \to X$. Suppose $f(\Delta)$ intersect transversely w/ D.

Have $f^*(K_X) \simeq \mathbb{C}^n$ on Δ.

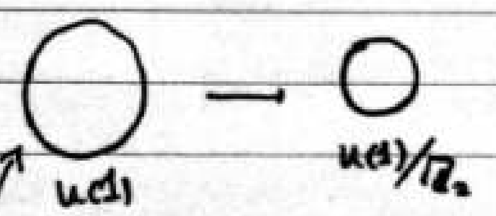
If $f^*\Omega$ is hol. on Δ, then $\mu(\beta) = 0$ since L is s.Lag. (the phase is const.)



Otherwise, $\left(\prod_i \frac{z-\alpha_i}{1-\bar{\alpha}_i z}\right)\Omega$ is hol. where α_i are the intersection pts w/ D.

Winding # of $\prod_i\left(\frac{z-\alpha_i}{1-\bar{\alpha}_i z}\right)\Omega$(vol. element of L) is $\beta \cdot D$,

$\therefore \mu(\beta) = 2(\beta \cdot D)$.



Loop on $U(1)$ winds on $U(1)/\mathbb{Z}_2$ twice.

Mirror space $= TP^\circ/\Lambda \subset (\mathbb{C}^\times)^n$.

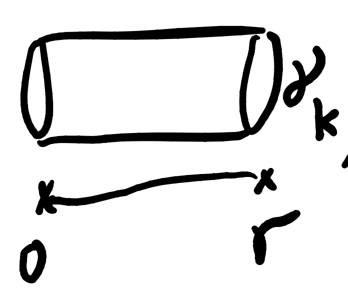
$\|$

$\{$flat $U(1)$ connections ∇ on fiber at $r \in P^\circ\}$

Complex coord. on mirror $z_k = \exp(-\underbrace{r_k} + i\theta_k)$

base action coord.

$= \exp(-\text{area}(\quad \gamma_k)) \cdot \text{Hol}_\nabla(\gamma_k)$.



For $\beta \in H_2(X, L_r)$,

$z^\beta \triangleq \exp(-\text{area}(\beta(r))) \cdot \text{Hol}_\nabla(\partial\beta)$

monomial in z_i.

Hol. discs in toric case

$\therefore$ In toric case, $\mu(\beta) = 2\beta \cdot (D_1 + \dots + D_m)$.

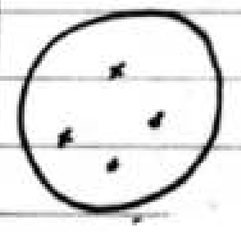
$\underbrace{\text{Hol. map } (\Delta, \partial\Delta) \to (X, L)}$ e.g. 

Maslov index 2

- Smooth hol. discs in a toric mfd is classified by [Cho-Oh]. (w.r.t. toric Käh str.)

e.g. $\mathbb{C}^N$. Consider $f : (\Delta, \partial\Delta) \xrightarrow[\text{hol.}]{(f_1 \dots f_N)} (\mathbb{C}^N, F_r)$.

$\Delta \supset f^{-1}(\underbrace{D_i}_{\text{toric div.}})$ must be isolated points by identity thm.

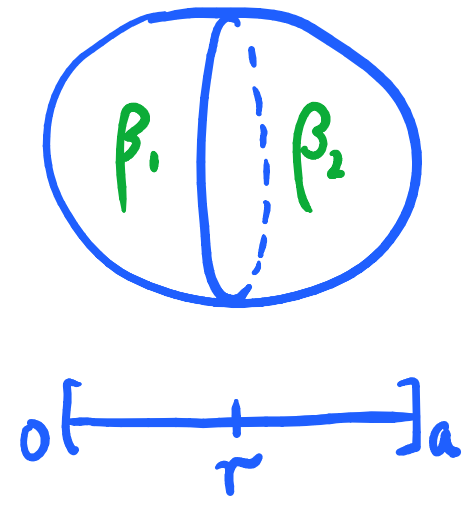


If f_i has no zeroes, then f_i is const. by max. principle. ($|f|$ is const)

Otherwise, inductively $\left(\prod_{j=1}^{l_i} \left(\frac{z - \alpha_j^i}{1 - \bar{\alpha}_j^i z}\right)\right)^{-1} \cdot f_i$ has no zero, and so is const c_i.

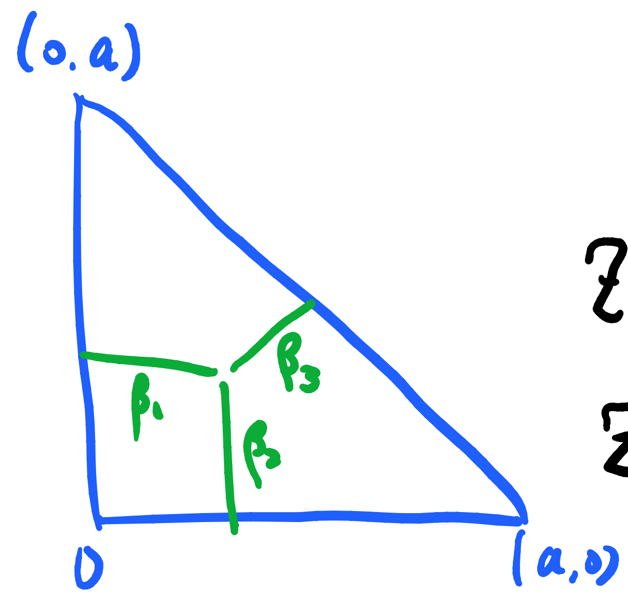
Blaschke factor. its norm is still const.

$$\therefore f_i = c_i \prod_{j=1}^{l_i} \frac{z - \alpha_j^i}{1 - \bar{\alpha}_j^i z}.$$

e.g. 

$$z^{\beta_1} = \exp(-\overbrace{\text{area}(\beta_1(r))}^{r})\, \overbrace{\text{Hol}_\nabla(\partial\beta_1)}^{\theta}$$
$$= \exp(-r + i\theta) = z.$$

$$z^{\beta_2} = \exp(-(a-r) - i\theta)$$
$$= \underbrace{\exp(-a)}_{q} z^{-1} = q z^{-1}.$$

e.g. 

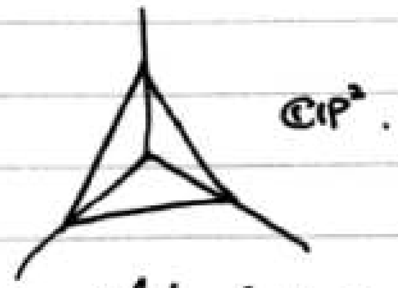
$$z^{\beta_1} = z_1.$$
$$z^{\beta_2} = z_2.$$

connecting to 0 is the same as connecting to any pt on the cor. divisor.

$$z^{\beta_3} = \exp(-(a - r_1 - r_2) - i(\theta_1 + \theta_2))$$
$$= \exp(-a) z_1^{-1} z_2^{-1} = \frac{q}{z_1 z_2}.$$

• $X_\Delta \simeq \mathbb{C}^m /\!/_{GIT} (\mathbb{C}^\times)^{m-n} \simeq \mathbb{C}^m /\!/_{sympl.} T^{m-n}$.

Preimage of $F_r \subset X_\Delta$ is a fiber of $\mathbb{C}^m \xrightarrow[(|z_i|^2)]{} \mathbb{R}^m_{\geq 0}$



Hol. disc bdd. by F_r is lifted to that bdd by $\mathbb{C}^m$.

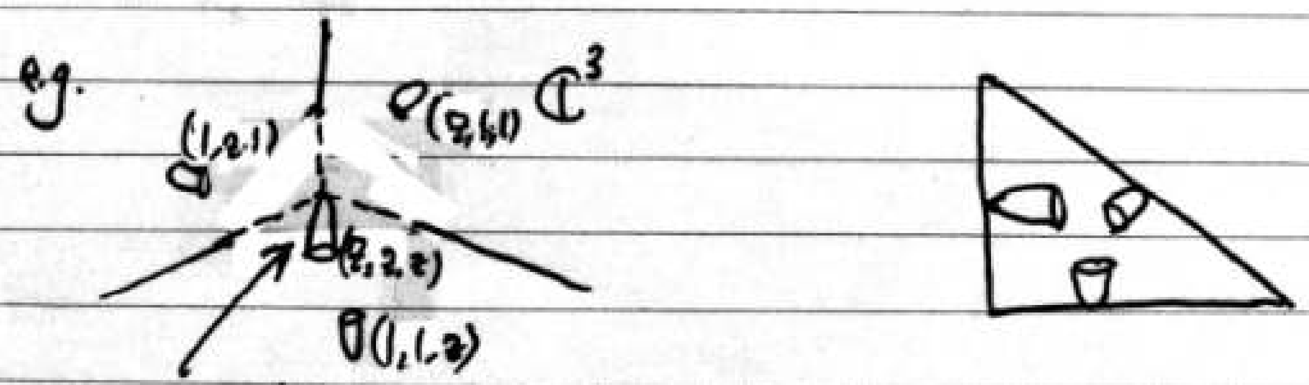
Thm. (Cho-Oh):

$f:(D,\partial D)\longrightarrow(X,F_r)$ must be of the form

$$\left[\tilde f = \left(c_1\prod_{j=1}^{\ell_1} \frac{z-\alpha_j^1}{1-\bar\alpha_j^1 z}, \dots\dots, c_m \prod_{j=1}^{\ell_m}\frac{z-\alpha_j^m}{1-\bar\alpha_j^m z}\right)\right]$$

s.t. $\operatorname{Im}\tilde f$ never intersects w/ the unstable locus $Z\subset\mathbb{C}^m$.

e.g.



this intersect w/ the unstable locus $\{0\}$ and so do not descend!

Basic disc class in $H^2(X, L_r)$:

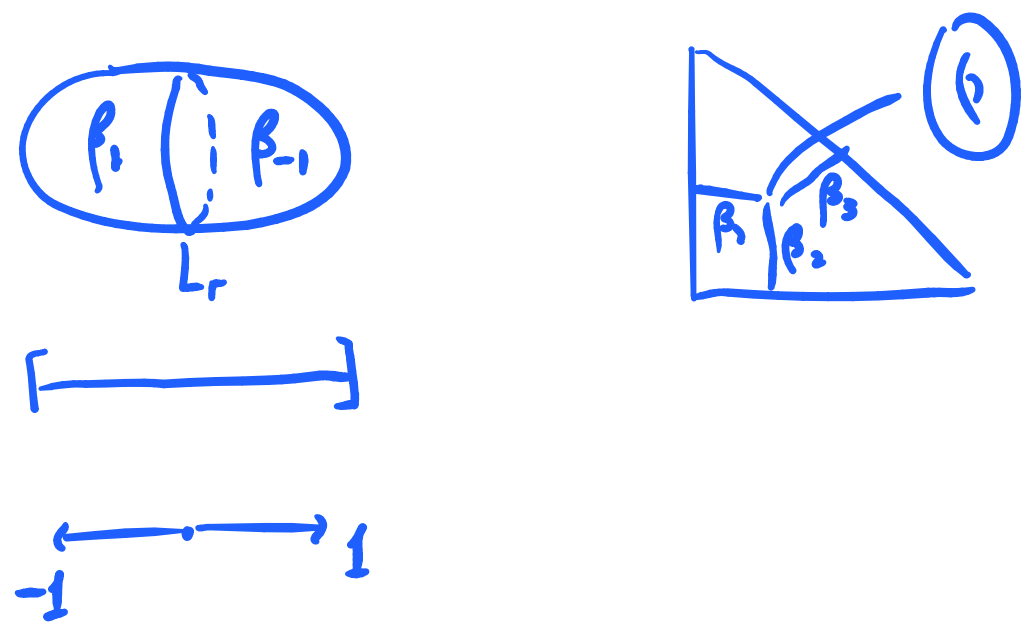
take v : prim. gen. of a ray of Σ.
$v \in N = \pi_1(L_r)$.

take a path connecting r to a point in D_v.
toric divisor

Move γ_v along the path

and γ_v vanishes at $D_v \longrightarrow$ disc β_v.

e.g.



$$W=\sum_v z^{\beta_v} \text{ for toric Fano mfd.}$$

- Maslov index of $\left[c_1 \prod_{j=1}^{l_1} \frac{z-\alpha_j^1}{1-\bar{\alpha}_j^1 z}, \ldots\ldots, c_m \prod_{j=1}^{l_m} \frac{z-\alpha_j^m}{1-\bar{\alpha}_j^m z}\right]$ is total # of Blaschke factor $= l_1 + \cdots + l_m$.
- Basic disc classes β_i rep. by $[c_1, \ldots, c_i z, \ldots, c_m]$.

  The classes of a hol. disc must be of the form $\sum_{i=1}^{m} l_i \beta_i$.
- The exact seq.

$$0 \to L \to \mathbb{Z}^m \to N \to 0$$

can be identified as

$$H_2(F_r) \to H_2(X) \to H_2(X_\Delta, F_r) \xrightarrow{\partial} H_1(F_r) \to \underbrace{H_1(X)}_{\overset{\|}{0} \text{ if } X_\Delta \text{ cpt}} \to \cdots$$

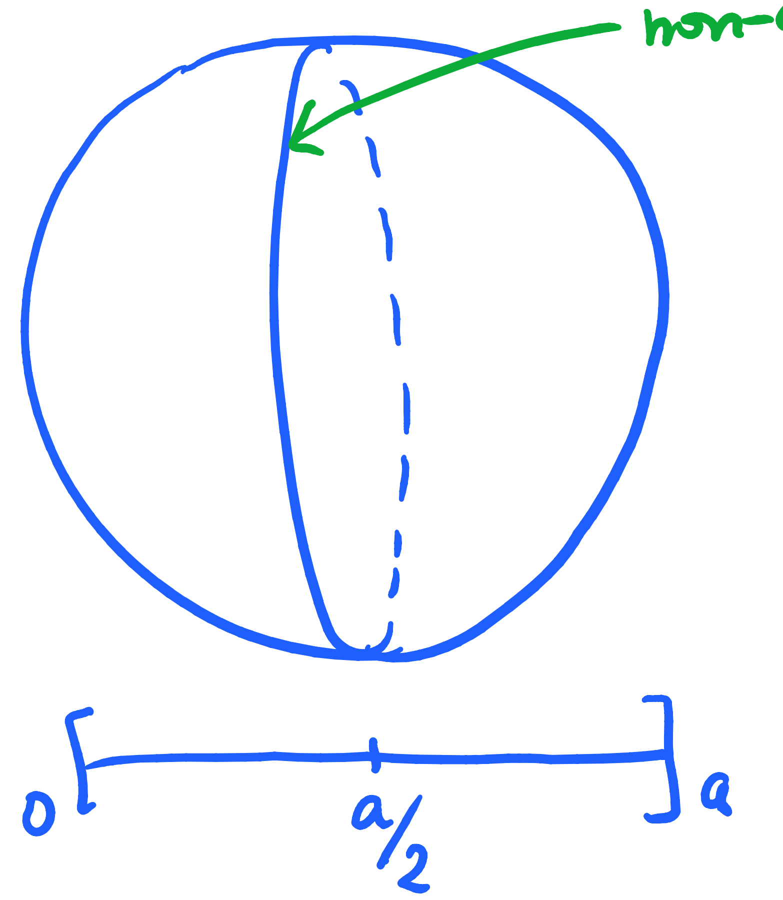
$$H_2(X_\Delta, F_r) \cong \mathbb{Z}\langle \beta_1 \ldots \beta_m \rangle, \qquad H_1(F_r) \cong N$$

0 map if X_Δ cpt.

Fukaya-Oh-Ohta-Ono:

Critical points of W corr. to non-displaceable fibers!

e.g.

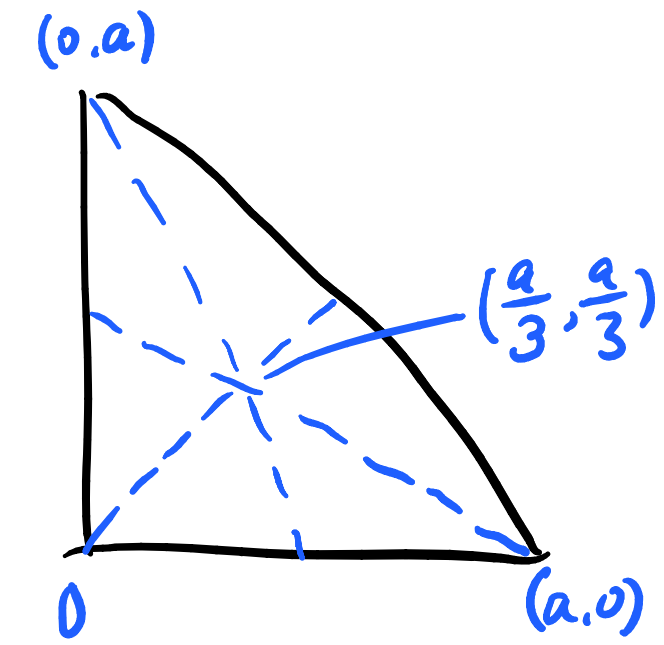


$$W = z + \frac{q}{z}.$$

$$\frac{\partial W}{\partial z} = 0 \iff z = \pm q^{1/2}$$

$$= \exp\left(-\frac{a}{2}\right) e^{i\theta}, \quad \theta = 0, \pi.$$

e.g.



$$W = z + w + \frac{q}{zw}.$$

crit. pts : $z = w = q^{1/3}$

$$= \exp\left(-\frac{a}{3}\right) e^{i\theta}, \quad \theta = 0, \pm\frac{2\pi}{3}.$$

Moduli of holomorphic discs. (to count discs)

• We always use the toric cpx. str. J.

The Lag. Fl. theory is indep of the choice of almost cpx strs (and perturbation in virtual theory).
(up to A_∞ isom.)

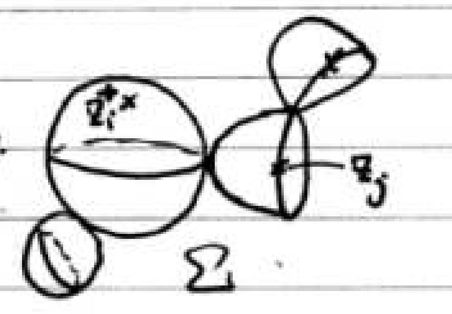
For $\beta \in H_2(X, L)$

$\partial\Sigma \ni$ mutually distinct $\Sigma \ni$

• $\mathcal{M}_{k+1,\ell}(\beta) \doteq \{(u : (\Sigma, \partial\Sigma) \longrightarrow (X, L),\ z_0, \dots, z_k,\ z_1^+, \dots, z_\ell^+)$ stable$\} / \mathrm{Aut}(\Sigma)$

* intersections are nodal.

e.g.



* J-hol. on each cpnt.

* $(z_0, \dots, z_k)$ are cyclically ordered.

* stable (aut finite) : If u is non-const on a cpnt,

then $2n_{int} + n_\partial \geq 3$ for disc cpnt

$n_{int} \geq 3$ for sphere cpnt.

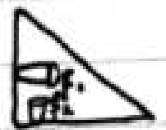
↳ #(int. marked or nodal pts).

• Have $ev : \mathcal{M}^{main}_{k+1,\ell}(\beta) \longrightarrow X^\ell \times L^{k+1}$.

Superpotential. (Fano)

- Recall $\quad T^*\Delta_0 / N \quad$ ⫴ $\quad T\Delta_0 / M = \check{X}$

$$\begin{array}{ccc} T^*\Delta_0/N & & T\Delta_0/M \\ \cap & & \Vert \\ X_\Delta & & \{(F_r, \nabla) : r \in \Delta_0 ;\ \nabla \text{ flat } U(1) \text{ conn. on } F_r\}. \\ \text{cpt toric} & & \end{array}$$

- WLOG. suppose $\{v_1, \dots v_n\}$ forms a cone in Σ. 

adjacent facets are coordinate hyperplanes

Cpx coord. on $\check{X}$ is

$$z_i = \underbrace{e^{-\int_{\beta_i}\omega}}_{e^{-r_i}} \underbrace{Hol_\nabla(\partial\beta_i)}_{e^{i\check{\theta}_i}}$$

- SYZ mirror obtained by packing n_β. $W^{LF} : \check{X} \longrightarrow \mathbb{C}$,

$$W^{LF} \triangleq \sum_{\beta \in H_2(X, F_r)} n_\beta\, e^{-\int_\beta \omega}\, Hol_\nabla(\partial\beta)$$

$$\underset{\text{Cho-Oh}}{=} \sum_{i=1}^{m} e^{-\int_{\beta_i}\omega}\, Hol_\nabla(\partial\beta_i).$$

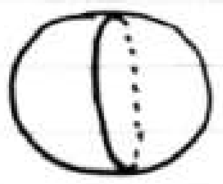
Using $\{v_1 \dots v_n\}$ as basis, $\partial\beta_i = \sum_{\ell=1}^{n} v_{i,\ell}\, \partial\beta_\ell$

(for $i = 1 \dots n$, $v_{i,\ell} = \delta_{i\ell}$.)

Then $\alpha_i \triangleq \beta_i - \sum_{\ell=1}^{n} v_{i\ell}\beta_\ell \in H_2(X)$. ($\alpha_i = 0$ for $i = 1 \dots n$).

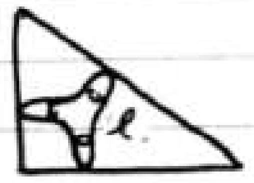
match w/ Hori-Vafa potential. $\rightarrow$ $$W^{LF} = \sum_{i=1}^{m} e^{-\int_{\alpha_i}\omega} \prod_{\ell=1}^{n} z_\ell^{v_{i\ell}} = \sum_{i=1}^{m} e^{-\int_{\alpha_i}\omega} z^{\vec{v}_i}.$$

e.g. $\mathbb{P}^1$. $W = z + \frac{q}{z}$.



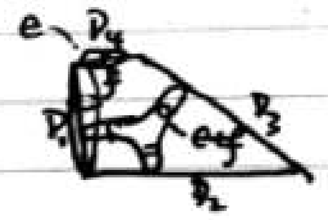
$q = e^{-\int_{\mathbb{P}^1} \omega}$.

e.g. $\mathbb{P}^2$. $W = z_1 + z_2 + \frac{q}{z_1 z_2}$



$q = e^{-\int_{\ell} \omega}$.

e.g. $\mathbb{F}^1$. $W = z_1 + z_2 + \frac{q^{e+f}}{z_1 z_2}$



$+ \frac{q^{e}}{z_2}$.

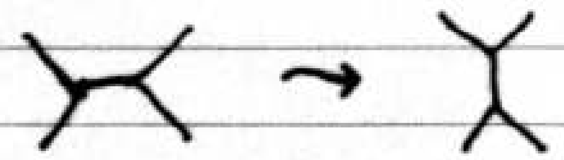
e.g. $\mathbb{P}^n$. $W = z_1 + \dots + z_n + \frac{q}{z_1 \cdots z_n}$.

For any toric variety,

$$W^{HV} \triangleq \sum_{i=1}^{m} e^{-\int_{\alpha_i} \omega} z^{\vec{v}_i}.$$

L depends only on rays of the fan.

(e.g. does not distinguish flop.)

loc. 

fan :