Quiver rep.

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Ch. 1 of [DW]

Quiver:
a directed graph where multiple edges and loops are allowed.

Quiver rep.:
attach each vertex x a vector space $V(x)$ and
each arrow a a linear map $V(a)$: $V(t_a) \to V(h_a)$.

Why are we interested?

- Start with simple linear algebra
- Have applications in other branches of sciences
- Deeply related with Lie theory
- Moduli theory involves interesting geometric constructions
- Also has interesting category and homological algebras
- Give a class of (comm. or non-comm.) geometries
- Efficient way to understand local CY geometries
- Arises in mirror symmetry and DT theory

Morphism ϕ: $V \to W$:
a linear map $\phi: V(x) \to W(x)$ for each $x \in Q_0$ such that
commuting with $V(a)$ and $W(a)$.

In other words,
$\phi \in C^0(V,W) = \oplus_x \operatorname{Hom}\big(V(x), W(x)\big)$ with
$d\phi = 0$, where

$$d:\ C^0(V,W) \to C^1(V,W) := \bigoplus_{a \in Q_1} \operatorname{Hom}\big(V(t_a), W(h_a)\big),$$

$$d\phi = \big(\phi(h_a)V(a) - W(a)\phi(t_a)\big).$$

After fixing basis of V and W, an isom. $\in GL_\alpha$.
Auto. is stabilizer of V under GL_α.

Have subrepresentations, kernel, cokernel, direct sum.

Representation space with dim. vector α:

$$Rep_\alpha(Q) := \prod_{a \in Q_1} \mathrm{Mat}_{\alpha(h_a),\alpha(t_a)} .$$

$$\phi \in GL_\alpha := \prod_{x \in Q_0} GL\big(\alpha(x)\big)$$

acts on $Rep_\alpha(Q)$ by
$(\phi \cdot V)(a) = \phi(h_a) \cdot V(a) \cdot \phi(t_a)^{-1}$.
Take $Rep_\alpha(Q)/GL_\alpha$.

Indecomposable and irreducible (same as simple) representations.

ex. Quiver with one vertex and one arrow.
Indecomposables are Jordan blocks.

Path algebra $\mathbb{C}Q$.

Quiver representations are equiv. to (left) $\mathbb{C}Q$-modules:
Module M gives rep. $V(x) = e_x M$;
Rep. V gives module $M = \bigoplus_x V(x)$.
($e_y \cdot V(x) = 0$ if $x \neq y$.)

Path algebra as "free algebra" $\mathbb{C}Q = TM$: (T means tensor)
From the $\mathbb{C} -$ bimod $M = \mathbb{C}^l$,
have free algebra (or tensor algebra) $TM.$
$TM = \mathbb{C}Q$ where Q is the quiver with one vertex and l arrows.

General quiver Q can be considered to be generalization of this:
Take the algebra $\mathbb{C}^n$ defined by
$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = (a_1 b_1, \dots, a_n b_n)$.
For $\mathbb{C}^n$ − bimod M,

$$M = \bigoplus_{i,j} e_j M e_i$$

where $e_i = (0, \dots 1, \dots, 0)$.
Take these to be vertices.
Choose a $\mathbb{C}$-basis of $e_j M e_i$ and define it to be arrows from i to j.
Get a quiver with $\mathbb{C}Q = TM$.

Krull-Remak-Schmidt Theorem:
Every f.d. quiver rep. is direct sum of indecomposables,
unique up to permutations.

Existence is trivial.
Uniqueness needs a "version of Schur's Lemma" below.

Jordan decomposition for endo. $\phi: V \to V$ of quiver rep.:

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$$

where V_λ is subrepresentation of generalized λ-eigenspaces.
$V_\lambda \coloneqq \{v \in V:\ (\phi - \lambda\, I)^N(v) = 0 \text{ for some } N\}$.
(Arrows preserve λ-generalized eigenspaces over vertices.)

Cor. 1.7.1.
If V indecomp., then exists unqiue λ such that
$(\phi - \lambda\, \mathrm{I})$ is nilpotent.

Now consider $\text{End}_Q(V)$, a ring.

Cor. 1.7.2. (analog of Schur's Lemma)
If V indecomp.,

$\phi \in \text{End}_Q(V)$ is either invertible or nilpotent.

$\mathfrak{m} := \{\phi \text{ not invertible}\} \subset \text{End}_Q(V)$
is the unique maximal ideal.

$\text{End}_Q(V)/\mathfrak{m} \cong \mathbb{C}$.

Proof:
There exists unique λ such that $(\phi - \lambda\, I)$ is nilpotent.
If $\lambda = 0$: ϕ nilpotent;
if $\lambda \neq 0$: $\phi = \lambda I + (\phi - \lambda I)$ invertible.

$\mathfrak{m}$ is the unique maximal ideal:
if $\phi \in J - \mathfrak{m}$, since ϕ is invertible, $J = \text{Hom}_Q(V, V)$.

$\text{End}_Q(V)/\mathfrak{m} \xrightarrow{\cong} \mathbb{C}$ by
$\phi \mapsto \lambda$ where λ is the unique number such that
$(\phi - \lambda\, I)$ is nilpotent.

Proof of uniqueness of decomposition:
Suppose $V_1 \oplus \cdots \oplus V_p \cong W_1 \oplus \cdots \oplus W_r$ by ϕ,
and let ψ be the inverse.

Write in block form:

$$\sum_i \psi_{p,i}\phi_{i,p} = I_{V_p} \notin \mathfrak{m}_p \subset \mathrm{End}_Q(V_p).$$

Then $\psi_{p,i}\phi_{i,p} \notin \mathfrak{m}_p$ for some i, say $i = r$.
$\psi_{p,r}\phi_{r,p}$ is invertible. So
$\phi_{r,p}$ is inj. and $\psi_{p,r}$ is surj.

$\phi_{r,p}\psi_{p,r}$ cannot be nilpotent, and hence is also invertible:
otherwise $\left(\psi_{p,r}\ \phi_{r,p}\right)^N = 0$.

Thus $\phi_{r,p}$ is iso. $V_p \cong W_r$.

Then ϕ gives an isomorphism
$V_1 \oplus \cdots \oplus V_{p-1} \cong W_1 \oplus \cdots \oplus W_{r-1}$:
Write ϕ in 2×2 matrix where one of the diagonal blocks is $\phi_{r,p}$.
Since $\phi_{r,p}$ is iso.,
can do row and column operations
(left and right multiply by invertibles)
to reduce to block diagonal matrix
(which is still invertible since ϕ is).
Done by induction.