Homology of quiver representations

Projective $A$-mod $P$: (cover)

\[
\begin{array}{c}
N \\ f \\
\downarrow \\
\rightleftharpoons \\
M' \leftarrow f \rightarrow M \\
\downarrow \\
I \\
\leftarrow h
\end{array}
\]

\[\Leftrightarrow \text{Hom}_A(P, -) \text{ exact}\]

Injective $I$: (contain)

\[
\begin{array}{c}
N \\ f \\
\downarrow \\
\rightleftharpoons \\
M' \leftarrow f \rightarrow M \\
\downarrow \\
I \\
\leftarrow h
\end{array}
\]

\[\Leftrightarrow \text{Hom}_A(-, I) \text{ exact}\]

Crucial to compute cohomology.
Projectives are so important that gradually you will use them even without remembering the def.! Just like the notion of "derivative".

Any exact $0 \to N \to M \to P \to 0$ or
$0 \to I \to M \to N \to 0$ split.
(Lift $\text{Id}: P \to P$ to $P \to M$.)

$M$ projective \[\Leftrightarrow M^* \text{ (dual v.s.) injective (as } A^{\text{op}} - \text{mod).}\]
$P' \oplus P''$ is projective.
Summand of $P$ is projective.

If $A$ is finite dimensional,
$A$ itself is a f.d. $A$-module.

Hom$_A(A, -) = Id$ and hence exact.
Thus $A$ is projective.

**Lemma 2.1.5.**

$P$ is a projective $A$-module $\Rightarrow P$ is a direct summand of $A^{\oplus r}$.
(Similar for injective $I$)

In particular if $P$ indecomp and $A$ is f.d.,
then $P$ is a direct summand of $A$.

**Proof.**
Take generators of $P$.
Then have $A^{\oplus r} \rightarrow P$ surjective, and hence split. QED

Now $A := \mathbb{C}Q$.
$P_x := A \cdot e_x$. ($l_x = (e_x \cdot A)^*$.)

$A = \bigoplus_x P_x$.
Thus $P_x$ are projective.

$P_x$ as representation: $P_x(y) = e_y \cdot A \cdot e_x$.

**Prop. 2.2.2.**
Hom$_Q(P_x, V) \cong V(x)$. Hom$_Q(V, I_x) \cong V(x)^{\ast}$.
\[ \phi \mapsto \phi(e_x) =: v. \]
\[ \phi \text{ is determined by its image: } \phi(p \cdot e_x) = V(p) \cdot v. \]

**Prop. 2.2.3.**
Assume $Q$ acyclic. (No directed cycle.)
$P_x$ are all the indecomp. projectives.
$I_x$ are all the indecomp. injectives.

**Proof.**
By previous prop,
Hom$_Q(P_x, P_x) \cong P_x(x) \cong \mathbb{C} \cdot e_x$ (acyclic).
Hence $P_x$ indecomp.
Suppose $P$ proj.
Lemma 2.1.5 $\Rightarrow$ $P$ is direct summand of $A^{\oplus r} = \bigoplus_x P_x^{\oplus r}$.
$P$ indecomp. forces $P = P_x$.

$S_x$: simple module over vertex $x$.
Proj. resolution:
\[
0 \to \bigoplus_{a: t_a = x} P_{h_a} \xrightarrow{a} P_x \to S_x \to 0.
\]
Inj. resolution:
\[
0 \to S_x \to I_x \to \bigoplus_{a: h_a = x} I_{t_a} \to 0.
\]
Given any module $V$, proj. resolution:
0 \to \bigoplus_a V(t_a) \otimes P_{ha} \xrightarrow{d} \bigoplus_x V(x) \otimes P_x \to V \to 0.

where \( d(v \otimes p)_a = ((a \cdot v) \otimes p)_{ha} - (v \otimes p \cdot a)_{ta}. \)

Obvious that it is a complex.

Exact:

Surjectivity of last map is obvious.

Injectivity of the first map:

Suppose non-zero and \( d \) to 0.

Consider the terms \((v \otimes p)_a\) that has the longest \( p \).

After \( d \), have \(- (v \otimes p \cdot a)_{ta}\) that cannot cancel with any other term.

Contradiction.

\( \text{Ker} \subset \text{Im} \) for the middle piece:

Fix each \( x \). If

\[
\sum_{h_p=x} v_{t(p)} \otimes p \text{ has } 0 = \sum_p p \cdot v_{t(p)} = cv_x + \sum_{p \not\in e_x} p \cdot v_{t(p)},
\]

Then

\[
\sum_p v_{t(p)} \otimes p = cv_x \otimes e_x + \sum_{p \not\in e_x} v_{t(p)} \otimes p
\]

\[
= - \sum_{p \not\in e_x} p \cdot v_{t(p)} \otimes e_x + \sum_{p \not\in e_x} v_{t(p)} \otimes p
\]

which belongs to the image of \( d \)
(by moving the arrows in \( p \) one by one).

\textbf{Remark:}

Injectivity above fails for quiver with relations.
**Cohomology** $H^i(V, W)$

Take the above projective resolution of $V$ (or injective resolution of $W$)

$0 \to P_1 \to P_0 \to 0.$

Get complex

$0 \to \text{Hom}_Q(P_0, W) \to \text{Hom}_Q(P_1, W) \to 0.$

(or $0 \to \text{Hom}_Q(V, I^0) \to \text{Hom}_Q(V, I^1) \to 0$.)

Take coho. of the complex, get $H^i(V, W)$ for $i = 0, 1.$

Recall the resolution of $V$ is

$$0 \to \bigoplus_a V(t_a) \otimes P_{h_a} \xrightarrow{d} \bigoplus_x V(x) \otimes P_x \to 0$$

where

$d(v \otimes p)_a = ((a \cdot v) \otimes p)_{h_a} - (v \otimes p \cdot a)_{t_a}.$

Take $\text{Hom}_Q(-, W)$.

For any v.s. $U$,

$\text{Hom}_Q(U \otimes P_x, W) = \text{Hom}(U, W(x)).$ (Prop. 2.2.2)

$C^0 = \text{Hom}_Q(P_0, W) = \bigoplus_x \text{Hom}(V(x), W(x))$

$C^1 = \text{Hom}_Q(P_1, W) = \bigoplus_a \text{Hom}(V(t_a), W(h_a))$

$0 \to C^0 \to C^1 \to 0.$

For $\phi \in \text{Hom}_Q(P_0, W), v \in V(t_a),$
\[(d\phi)\left((v \otimes e_{h_a})_a\right) = \phi\left(d\left((v \otimes e_{h_a})_a\right)\right)\]
\[= \phi\left((V(a) \cdot v) \otimes e_{h_a}\right)_{h_a} - \phi(v \otimes a)_{t_a}\]

which is identified as \(\phi(h_a)V(a) \cdot v - W(a)\phi(t_a)(v)\).

Thus \(d\phi = \phi(h_a)V(a) - W(a)\phi(t_a)\)

which is defined previously (measuring commutativity).

\[H^0 = \text{Ker}(d) = \text{Hom}_Q(V, W).\]

\[H^1 = \text{coKer}(d):\]

Given \(\psi \in C^1(V, W)\), define a corresponding rep. \(E = E_\psi\),

\[E(x) = V(x) \oplus W(x);\]

\[E(a) = \begin{pmatrix} V(a) & 0 \\ \psi(a) & W(a) \end{pmatrix}.\]

Have exact sequence
\[0 \to W \to E_\psi \to V \to 0.\]

(called extension of \(V\) by \(W\))

Any extension must be of this form (vector space is classified by its dimension).

Suppose \(\psi - \psi' = d\phi\).

Then have
\[0 \to W \to E_\psi \to V \to 0\]
\[\downarrow \text{Id} \downarrow \text{Id} \downarrow \text{Id}\]
\[0 \to W \to E_{\psi'} \to V \to 0.\]

Middle arrow (called equivalence of extensions) is
\[
\begin{pmatrix}
\text{Id} & 0 \\
-\phi(x) & \text{Id}
\end{pmatrix}.
\]

\(\psi - \psi' = d\phi \iff\) commutative diagram.

Thus cokernel is set of extensions up to equivalence.
Prop. 2.5.2.
Euler char.
\[ \chi(V,W) = \langle \alpha, \beta \rangle := \sum_x \alpha(x)\beta(x) - \sum_a \alpha(t_a)\beta(h_a) \]
where \( \alpha, \beta \) are dim. vectors of \( V, W \).

Proof.
Exact:
\[ 0 \to H^0 \to \bigoplus_x \text{Hom}(V(x), W(x)) \to \bigoplus_a \text{Hom}(V(t_a), W(h_a)) \to H^1 \to 0. \]

Euler matrix
\[ E_{i,j} = \delta_{i,j} - |\text{arrows from } i \text{ to } j|. \]

Hereditary algebra:
Submod. of projective module is projective.
(Always true for summand)

Thm. 2.3.2.
\( \mathbb{C}Q \) is hereditary (where \( Q \) acyclic).

Proof.
Let \( M \subset P \). Get exact \( 0 \to M \to P \to V \to 0 \).
By above have proj. resol.
\[ 0 \to P_1 \to P_0 \to V \to 0. \]
The following prop. implies that \( M \) is proj. QED

Prop. 2.3.3.
Given two exact sequences as above (where $P, P_0, P_1$ are proj.),
$P \oplus P_1 \cong P_0 \oplus M$.

**Proof.**
P $\to V$ can be lifted to $P \to P_0$. Then get the chain map:

$$0 \to M \to P \to V \to 0$$

Can use the first square to define a complex
$$0 \to M \to P \oplus P_1 \to P_0 \to 0.$$  
First map is injective: trivial.
Last map is surjective by diagram chasing.
Dimension count implies this is exact.
Since $P_0$ projective, the sequence splits.