Homology of quiver representations

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Ch. 2 of [DW]

Projective A-mod P: (cover)

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "g"] \\ N \arrow[r, two heads, "f"'] & N' \end{tikzcd}$$

$\Leftrightarrow \mathrm{Hom}_A(P, -)$ exact

Injective I: (contain)

$$\begin{tikzcd} M' \arrow[r, hook, "f"] \arrow[d, "g"'] & M \arrow[dl, dotted, "h"] \\ I & \end{tikzcd}$$

$\Leftrightarrow \mathrm{Hom}_A(-, I)$ exact

Crucial to compute cohomology.
Projectives are so important that gradually you will use them even without remembering the def.! Just like the notion of "derivative".

Any exact $0 \to N \to M \to P \to 0$ or
$0 \to I \to M \to N \to 0$ split.
(Lift $Id: P \to P$ to $P \to M$.)

M projective $\Leftrightarrow M^*$ (dual v.s.) injective (as $A^{op} - \text{mod}$).

$P' \oplus P''$ is projective.
Summand of P is projective.

If A is finite dimensional,
A itself is a f.d. A-module.

$\mathrm{Hom}_A(A, -) = Id$ and hence exact.
Thus A is projective.

Lemma 2.1.5.
P is a projective A-module $\Rightarrow$ P is a direct summand of $A^{\oplus r}$.
(Similar for injective I)

In particular if P indecomp and A is f.d.,
then P is a direct summand of A.

Proof.
Take generators of P.
Then have $A^{\oplus r} \to P$ surjective, and hence split. QED

Now $A \coloneqq \mathbb{C}Q$.
$P_x \coloneqq A \cdot e_x$. $(I_x = (e_x \cdot A)^*.)$

$$A = \bigoplus_x P_x \,.$$

Thus P_x are projective.

P_x as representation: $P_x(y) = e_y \cdot A \cdot e_x$.

Prop. 2.2.2.

$\mathrm{Hom}_Q(P_x, V) \cong V(x)$. $\mathrm{Hom}_Q(V, I_x) \cong V(x)^*$.
$\phi \mapsto \phi(e_x) =: v$.
ϕ is determined by its image: $\phi(p \cdot e_x) = V(p) \cdot v$.

Prop. 2.2.3.
Assume Q acyclic. (No directed cycle.)
P_x are all the indecomp. projectives.
I_x are all the indecomp. injectives.

Proof.
By previous prop,
$\mathrm{Hom}_{\mathrm{Q}}(P_x, P_x) \cong P_x(x) \cong \mathbb{C} \cdot e_x$ (acyclic).
Hence P_x indecomp.
Suppose P proj.
Lemma 2.1.5 $\Rightarrow$ P is direct summand of $A^{\oplus r} = \bigoplus_x P_x^{\oplus r}$.
P indecomp. forces $P = P_x$.

S_x: simple module over vertex x.
Proj. resolution:

$$0 \to \bigoplus_{a:\, t_a = x} P_{h_a} \xrightarrow{a} P_x \to S_x \to 0.$$

Inj. resolution:

$$0 \to S_x \to I_x \to \bigoplus_{a: h_a = x} I_{t_a} \to 0.$$

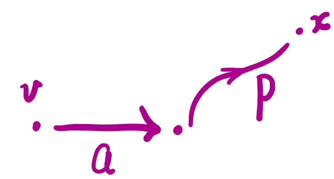
Given any module V, proj. resolution:

$$0 \to \bigoplus_a V(t_a) \otimes P_{h_a} \xrightarrow{d} \bigoplus_x V(x) \otimes P_x \to V \to 0.$$

where $d(v \otimes p)_a = \big((a \cdot v) \otimes p\big)_{h_a} - (v \otimes p \cdot a)_{t_a}$.

Obvious that it is a complex.
Exact:
Surjectivity of last map is obvious.
Injectivity of the first map:

Suppose non-zero and d to 0.
Consider the terms $(v \otimes p)_a$ that has the longest p.
After d, have $-(v \otimes p \cdot a)_{t_a}$ that cannot cancel with any other term.
Contradiction.



Ker $\subset$ Im for the middle piece:

Fix each x. If

$$\sum_{h_p = x} v_{t(p)} \otimes p \text{ has } 0 = \sum_p p \cdot v_{t(p)} = c v_x + \sum_{p \neq e_x} p \cdot v_{t(p)},$$

Then

$$\sum_p v_{t(p)} \otimes p = c v_x \otimes e_x + \sum_{p \neq e_x} v_{t(p)} \otimes p$$

$$= -\sum_{p \neq e_x} p \cdot v_{t(p)} \otimes e_x + \sum_{p \neq e_x} v_{t(p)} \otimes p$$

which belongs to the image of d
(by moving the arrows in p one by one).

Remark:
Injectivity above fails for quiver with relations.

Cohomology $H^i(V,W)$

Take the above projective resolution of V (or injective resolution of W)

$0 \to P_1 \to P_0 \to 0.$

Get complex

$0 \to \mathrm{Hom}_Q(P_0,W) \to \mathrm{Hom}_Q(P_1,W) \to 0.$

(or $0 \to \mathrm{Hom}_Q(V,I^0) \to \mathrm{Hom}_Q(V,I^1) \to 0.$)

Take coho. of the complex, get $H^i(V,W)$ for $i = 0,1$.

Recall the resolution of V is

$$0 \to \bigoplus_a V(t_a) \otimes P_{h_a} \xrightarrow{d} \bigoplus_x V(x) \otimes P_x \to 0$$

where

$$d(v \otimes p)_a = \big((a \cdot v) \otimes p\big)_{h_a} - (v \otimes p \cdot a)_{t_a}.$$

$\overset{v}{\cdot} \xrightarrow{a} \cdot \xrightarrow{p}$

Take $\mathrm{Hom}_Q(-,W)$.

For any v.s. U,

$\mathrm{Hom}_Q(U \otimes P_x, W) = \mathrm{Hom}\big(U, W(x)\big)$. (Prop. 2.2.2)

$$C^0 = \mathrm{Hom}_Q(P_0,W) = \bigoplus_x \mathrm{Hom}\big(V(x),W(x)\big)$$

$$C^1 = \mathrm{Hom}_Q(P_1,W) = \bigoplus_a \mathrm{Hom}\big(V(t_a),W(h_a)\big)$$

$0 \to C^0 \to C^1 \to 0.$

For $\phi \in \mathrm{Hom}_Q(P_0,W)$, $v \in V(t_a)$,

$$(d\phi)\left(\left(v \otimes e_{h_a}\right)_a\right) = \phi\left(d\left(\left(v \otimes e_{h_a}\right)_a\right)\right)$$

$$= \phi\left((V(a) \cdot v) \otimes e_{h_a}\right)_{h_a} - \phi(v \otimes a)_{t_a}$$

which is identified as $\phi(h_a)V(a) \cdot v - W(a)\phi(t_a)(v)$.
Thus $d\phi = \phi(h_a)V(a) - W(a)\phi(t_a)$
which is defined previously (measuring commutativity).

$H^0 = \mathrm{Ker}(d) = \mathrm{Hom}_Q(V, W)$.

$H^1 = \mathrm{coKer}(d)$:

Given $\psi \in C^1(V, W)$, define a corresponding rep. $E = E_\psi$,
$E(x) = V(x) \oplus W(x)$;
$E(a) = \begin{pmatrix} V(a) & 0 \\ \psi(a) & W(a) \end{pmatrix}$.
Have exact sequence
$0 \to W \to E_\psi \to V \to 0$.
(called extension of V by W)
Any extension must be of this form (vector space is classified by its dimension).

Suppose $\psi - \psi' = d\phi$.
Then have

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r] \arrow[d, "Id"] & E_\psi \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, "Id"] & 0 \\ 0 \arrow[r] & W \arrow[r] & E_{\psi'} \arrow[r] & V \arrow[r] & 0. \end{tikzcd}$$

Middle arrow (called equivalence of extensions) is
$\begin{pmatrix} Id & 0 \\ -\phi(x) & Id \end{pmatrix}$.
$\psi - \psi' = \mathrm{d}\phi \Leftrightarrow$ commutative diagram.
Thus cokernel is set of extensions up to equivalence.

Prop. 2.5.2.
Euler char.

$$\chi(V,W) = \langle \alpha, \beta \rangle := \sum_x \alpha(x)\beta(x) - \sum_a \alpha(t_a)\beta(h_a)$$

where α, β are dim. vectors of V, W.

Proof.
Exact:

$$0 \to H^0 \to \bigoplus_x \mathrm{Hom}\big(V(x), W(x)\big) \to \bigoplus_a \mathrm{Hom}\big(V(t_a), W(h_a)\big)$$

$\to H^1 \to 0.$

Euler matrix
$E_{i,j} = \delta_{ij} - |\text{arrows from } i \text{ to } j|.$

Hereditary algebra:
Submod. of projective module is projective.
(Always true for summand)

Thm. 2.3.2.
$\mathbb{C}Q$ is hereditary (where Q acyclic).

Proof.
Let $M \subset P$. Get exact $0 \to M \to P \to V \to 0$.
By above have proj. resol.
$0 \to P_1 \to P_0 \to V \to 0$.
The following prop. implies that M is proj. QED

Prop. 2.3.3.

Given two exact sequences as above
(where P, P_0, P_1 are proj.),
$P \oplus P_1 \cong P_0 \oplus M$.

Proof.
$P \to V$ can be lifted to $P \to P_0$. Then get the chain map:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f_1"] & P \arrow[r] \arrow[d, "f_0"] & V \arrow[r] \arrow[d, "Id"] & 0 \\ 0 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & V \arrow[r] & 0 \end{tikzcd}$$

Can use the first square to define a complex
$0 \to M \to P \oplus P_1 \to P_0 \to 0$.
First map is injective: trivial.
Last map is surjective by diagram chasing.
Dimension count implies this is exact.
Since P_0 projective, the sequence splits.