Main Thm 4.4.13:
Finite representation type (only has finitely many indecomposable representations up to iso.)
⇔ the underlying graph is ADE Dynkin diagram.

In such a case, taking dimension vector gives a bijection
\{Indecomposable representations\} ↔ \{Positive roots\}.

⇒)
Use **Tits quadratic form** (on dim. vectors) (Analog of Killing form)

\[ B(\alpha) := \langle \alpha, \alpha \rangle = \sum_{x \in Q_0} \alpha(x)^2 - \sum_{a \in Q_1} \alpha(t_a)\alpha(h_a) \]

= \dim \text{End}_{\alpha} - \dim \text{Rep}_{\alpha}(Q).
(Does not depend on arrow directions)

Lemma 4.1.3:
Finite representation type
⇒ \(B(\alpha) \geq 1 \ \forall \alpha \neq 0\) integral positive (*)&
⇒ \(B\) is positive definite.

Proof.
\(GL_{\alpha}\) acts on \(\text{Rep}_{\alpha}(Q)\).
Only finitely many orbits ⇒ one of the orbits is open.
\(\dim(\text{that orbit}) = \dim \text{Rep}_{\alpha}(Q) = \dim GL_{\alpha} - \dim \text{stabilizer}\).
dim stabilizer ≥ 1 since has overall scaling.
Thus \(B(\alpha) \geq 1\).

Lemma 4.2.3.
If (*) holds for the quiver, then it holds for any subquiver (extend the dimension vector by zero, and \(B_{Q'}(\alpha) \geq B_{\bar{Q}}(\bar{\alpha})\)).
Lemma 4.2.1.

For extended ADE, has explicit \( \beta \) with \( B(\beta) = 0 \).

Thus finite representation type implies cannot contain extended Dynkin quiver.
Then it must be trivalent tree and of ADE type.

\[ \leftarrow \) ADE are finite representation type.
Can check directly that \( B(\alpha) \geq 1 \forall \alpha \neq 0 \) integral positive.

`Weyl reflection' on ANY quiver

to reduce indecomposable representation (positive root) to
simple representation (simple root) over vertices (of the new quiver).
(Underlying graph remains the same under reflection.)

Reflection functor \( C_x : Rep(Q) \to Rep(Q') \) at a sink or source vertex \( x \):
\( Q' \): reverse the adjacent arrows of \( x \).
For rep. \( V \):
\begin{itemize}
- Sink: replace the vector space at that vertex by sum of kernels of the adjacent arrows. (And the reversed arrows are the inclusions of
projections.)
Source: replace by cokernels.
For morphism \( \phi: V \to V' \):
Reflected rep.: \( W, W' \).
\( \psi(x): W(x) \to W(x)' \): restriction of \( \phi \) on kernels of adjacent arrows
(\( \phi \) maps kernel to kernel').
(Or induced map on cokernels of adjacent arrows.
\( \phi \) maps image to image')
For other vertices \( y, \psi(y) = \phi(y) \).
Note: taking \( C_x \) does not lose any info. if surj. at sink \( x \) or inj. at source \( x \).

Split away the simple \( S_x \):

**Lem. 4.3.6.**
\( x \) sink: \( S_x \) is a direct summand of \( V \) iff the sum of adjacent arrows is not surjective.
\( x \) source: \( S_x \) is a direct summand of \( V \) iff the sum of adjacent arrows is not injective.

**Proof.**
Choose a complement of \( \text{Im} \subset V(x) \) and get \( S_x^{\oplus k} \).
(or \( S_x^{\oplus k} \) is Ker for source)
Replacing \( V(x) \) by \( \text{Im} \) gives \( V' \).
(or replace by complement of \( \text{Ker} \).)
\( V = V' \oplus S_x^{\oplus k} \).

Reflection "almost" sends indecomp. to indecomp.:

**Thm. 4.3.9. [Bernstein-Gelfand-Ponomarev]**
Have natural transformation
\( (x \text{ sink}) \ C_x^2 \to Id \)
\( (x \text{ source}) \ Id \to C_x^2 \).
1. If \( V \) indecomposable, \( C_x(V) \) is indecomp. and \( C_x^2(V) \cong V \), UNLESS \( V \cong S_x \).
2. \( V \cong S_x \iff C_x(V) = 0 \).

In Case 1,
Lem. 4.3.6 \( \Rightarrow \) adjacent arrows are surj. (for sink) or inj. (for source).
dim. vector becomes
\[ \sigma_x(\alpha) \big|_x = \sum_{y \text{ adj. to } x} \alpha(y) - \alpha(x) \]

and remains the same for other vertices.

In Case 2, for the same def. of \( \sigma \), \( \sigma_x(\alpha) \big|_x = -1 \).

Hence can tell which case by considering \( \sigma_x(\dim V) \).

**Proof:**

\( x \) sink:

\[ C^2_x(V) \big|_x \text{ equals to } V(in_x)/\text{Ker}, \text{ and equals to } V(y) \text{ on other vertices } y. \]

Thus have natural morphism \( C^2_x(V) \to V \)

which is the induced map \( V(in_x)/\text{Ker} \to V(x) \) at \( x \), and Id on other vertices.

\( x \) source:

\[ C^2_x(V) \big|_x \text{ equals to image of sum of adjacent arrows.} \]

Thus have natural morphism \( V \to C^2_x(V) \)

which is the arrow map \( V(x) \to \text{Im} \) at \( x \).

Case 2 \( V \cong S_x \) is easy computation (of kernel and cokernel).

Case 1 \( V \not\cong S_x \): Lem. 4.3.6 \( \Rightarrow V(in_x)/\text{Ker} \cong V(x) \) (surj) or \( V(x) \cong \text{Im} \) (inj).

If \( C_x(V) \) is decomp., so is \( V \cong C^2_x(V) \), a contradiction.

**Strategy:**

Given an indecomposable, keep on reflecting, until it get to zero. Then the representation right before zero is simple.

Key: need to do it systematically to ensure it get to zero.

Keep track by dimension vectors (which is positive before it gets to zero).

Then Coxeter element should send the dim. vector to neg. (get out of first quadrant)

**Weyl group**

\( \sigma_x: \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0} \) with \( \sigma_x^2 = \text{Id} \). (\( x \) does not need to be sink nor source.)

\( W \): group generated by \( \sigma_x, x \in Q_0. \)

"Root": elements in a \( W \)-orbit of \( e_i. \)

**Prop. 4.4.9.** For ADE quiver,

1. Tits form \( B \) is pos. def. and preserved by \( W \).
2. \( B(\alpha) = 1 \) for root \( \alpha \).
3. Only finitely many roots.
4. \( W \) is finite.
5. For root \( \alpha \), either \( \alpha \geq 0 \) or \( \alpha \leq 0 \).

**Proof.**
1 is by direct check.
2 follows from \( B \) being \( W \)-inv.
3 is by compactness of \( \{ B = 1 \} \cap \mathbb{Z}^n \).
4 is \( W \subset \text{Perm} \text{(roots)} \).
5: Write \( \alpha = \alpha^+ - \alpha^- \) where \( \alpha^+(x) = \alpha(x) \) if positive, \( \alpha^-(x) = -\alpha(x) \) if neg.

1 = \( B(\alpha) = B(\alpha^+) + B(\alpha^-) - 2\langle \alpha^+, \alpha^- \rangle \geq B(\alpha^+) + B(\alpha^-) \)
since \( \langle \alpha^+, \alpha^- \rangle = \sum_x \alpha^+(x)\alpha^-(x) - \sum_a \alpha^+(t_a)\alpha^-(h_a) = -\sum_a \alpha^+(t_a)\alpha^-(h_a) \leq 0 \).
Then either \( B(\alpha^+) = 0 \) or \( B(\alpha^-) = 0 \) since \( B \) is pos. def. and integral.

**Def. 4.4.10. Coxeter element:**
\( c = \sigma_{x_n} \ldots \sigma_{x_1} \) product of all generators from base. (Choose an order.)

**Lem. 4.4.12.** Send to negative:
\( c: \mathbb{R}^{Q_0} \to \mathbb{R}^{Q'_0} \) has no fixed point other than 0.
Moreover, for any \( \alpha \neq 0 \), \( c^k(\alpha) \) has neg. coord. for some \( k \geq 0 \).

**Coxeter functor:**
To realize for quiver, need to choose correct order so that vertices involved are sinks.

**Def. 4.4.1.**
Take a seq. of vertices \( x_1, \ldots, x_m \) such that \( x_{i+1} \) is sink in \( Q_i = \sigma_{x_i} \ldots \sigma_{x_1}(Q) \).
(In particular \( x_i \neq x_{i+1} \).)
\( C^+ := C_{x_m} \ldots C_{x_1}: \text{Rep}(Q) \to \text{Rep}(Q') \);
\( C^- := C_{x_1} \ldots C_{x_m}: \text{Rep}(Q') \to \text{Rep}(Q) \). (Turn sources back to sinks.)

**Lem. 4.4.3. Coxeter functor for trees.**
Have such a sequence of sinks for trees with \( m = |Q_0| \).
(Order it with \( h_a < t_a \). \( (1, \ldots, n) \) gives such a sequence.)
**Lem. 4.4.4.**
Two quivers with the same underlying tree are related by \( Q' = \sigma_{x_m} \ldots \sigma_{x_1}(Q) \) for some sequence of sinks.
(Proof by induction.)

**Lem. 4.4.2.**

\( \text{Ind}(Q) \): set of indecomposables.
\[
M := \left\{ C_{x_1}^\text{source} \ldots C_{x_i}^\text{source} \left( S_{x_i}^\text{sink} \in \text{Ind}(Q_{i-1}) \right) \in \text{Rep}(Q) : i = 1, \ldots, m \right\}.
\]
\[
M' := \left\{ C_{x_i}^\text{sink} \ldots C_{x_{i+1}}^\text{sink} \left( S_{x_i}^\text{source} \in \text{Ind}(Q_i) \right) \in \text{Rep}(Q') : i = 1, \ldots, m \right\}.
\]

\( \text{Ind}(Q) - M \cong \text{Ind}(Q') - M' \) by \( C^\pm \) (inverse to each other).
\( M \to \{0\}, M' \to \{0\} \).
Immediately follows from Thm. 4.3.9.

**Ex. 4.4.2.**
Let \( x_1 \ldots x_m \) distinct.
\[
C_{x_1}^\text{source} \ldots C_{x_i}^\text{source} \left( S_{x_i}^\text{sink} \in \text{Ind}(Q_{i-1}) \right) = \mathbb{C}Q \cdot e_i.
\]
\[
C_{x_i}^\text{sink} \ldots C_{x_{i+1}}^\text{sink} \left( S_{x_i}^\text{source} \in \text{Ind}(Q_i) \right) = (e_i \cdot \mathbb{C}Q')^*.
\]
Note that \( S_{x_i}^\text{sink} = \mathbb{C}Q \cdot e_i \) since \( x_i \) is sink.
If reflect at adjacent source \( y \), \( C_y^\text{source}(S_{x_i}^\text{sink}) = \mathbb{C}\tilde{Q} \cdot e_i \) by def. (replacing 0 over \( y \) by cokernel which is \( \mathbb{C} \)).
Do induction.

**Def. 4.4.7. Coxeter functor:**
Given a sequence of sinks \( x_1 \ldots x_n \) where \( n = |Q_0| \).
The corresponding \( C^+, C^- : \text{Rep}(Q) \to \text{Rep}(Q) \) are called Coxeter functors.
(each arrow is adjacent to two vertices and hence reversed twice. Thus \( Q' = Q \)).

By above exercises and Lem. 4.4.2,
\( M \) consists of all proj. indecomp. and \( M' \) consists of all inj. decomp.

**Cor. 4.4.8.**
\( V \) proj. \( \iff \) \( C^+(V) = 0 \).
\( V \) inj. \( \iff \) \( C^-(V) = 0 \).
\( V \) has no proj. summands \( \Rightarrow C^-C^+(V) \cong V \).
\( V \) has no inj. summands \( \Rightarrow C^+C^-(V) \cong V \).
(Think about indecomposables first)

**Cor. 4.4.11.**
If \( V \) has no proj. summand, 
\( C^+(V) \) is non-zero rep. with dim. \( c(\alpha) \) (where \( c = \sigma_n \ldots \sigma_1 \) and \( \alpha \) is \( \dim V \)).

**Lem. 4.4.6.** Independent of order:
\( C^+ \) are naturally equivalent under (admissible) reordering of \( x_i \).

**Thm. 4.4.13.**
If ADE, then \( Ind(Q) \cong \{ \text{pos. roots} \} \) by \( V \mapsto \dim V \).
In particular finite.

**Proof:**
For \( V \in Ind(Q) \),  
\( c^k(\dim V) \) is no longer positive for some \( k > 0 \).
Take min. such \( k \).
Then \( C^+ \) is non-zero.
Also \( \sigma_j \ldots \sigma_1 c^{k-1}(\dim V) \) is no longer positive for some \( j > 0 \),  
take min. such \( j \).
Then \( C_{j-1} \ldots C_1 C^+ \) is non-zero and \( C_j \left( C_{j-1} \ldots C_1 C^+ \right) = 0 \).
Hence \( C_{j-1} \ldots C_1 C^+ \) is a positive root by def.

Given pos. root \( \alpha \), do the same thing to get 
\( \sigma_{j-1} \ldots \sigma_1 c^{k-1}(\alpha) \) positive but not \( \sigma_j \ldots \sigma_1 c^{k-1}(\alpha) \).
Then \( \sigma_{j-1} \ldots \sigma_1 c^{k-1}(\alpha) = \epsilon_j \).
(Recall that for root, coord. either all pos or all neg.)
Taking \( C^{-k-1} C_1 \ldots C_{j-1}(S_j) \) gives an indecomp. rep.