Gabriel's Theorem

Monday, December 2, 2019 11:07 AM

Ch. 4 of [DW]

Main Thm 4.4.13:

Finite representation type (only has finitely many indecomposable representations up to iso.)

$\Leftrightarrow$ the underlying graph is ADE Dynkin diagram.

In such a case, taking dimension vector gives a bijection

{Indecomposable representations} $\leftrightarrow$ {Positive roots}.

$\Rightarrow$)

Use **Tits quadratic form** (on dim. vectors) (Analog of Killing form)

$$B(\alpha) := \langle \alpha, \alpha \rangle = \sum_{x \in Q_0} \alpha(x)^2 - \sum_{a \in Q_1} \alpha(t_a)\alpha(h_a)$$

$= \dim \mathrm{End}_\alpha - \dim Rep_\alpha(Q)$.

(Does not depend on arrow directions)

Lemma 4.1.3:

Finite representation type

$\Rightarrow$ $\mathrm{B}(\alpha) \geq 1$ $\forall \alpha \neq 0$ integral positive (*)

$\Rightarrow$ B is positive definite.

Proof.

GL_α acts on $Rep_\alpha(Q)$.

Only finitely many orbits $\Rightarrow$ one of the orbits is open.

dim(that orbit) $= \dim Rep_\alpha(Q) = \dim GL_\alpha - \dim \text{stabilizer}$.

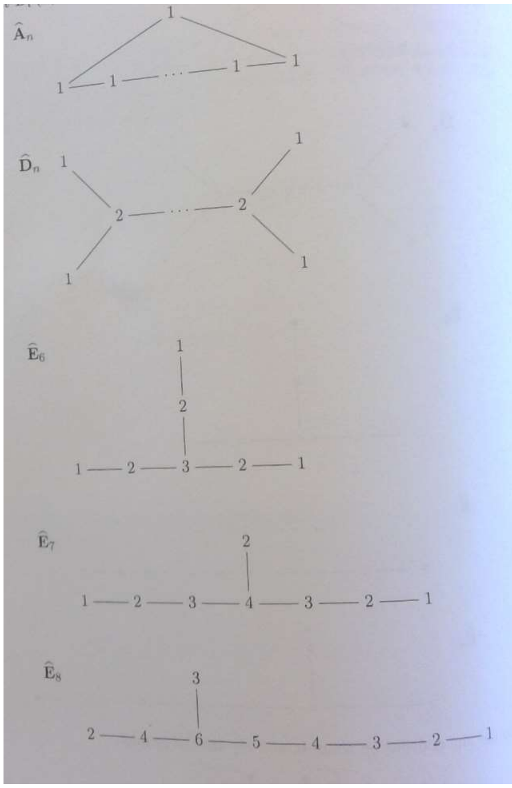
dim stabilizer ≥ 1 since has overall scaling.

Thus $B(\alpha) \geq 1$.

Lemma 4.2.3.

If (*) holds for the quiver, then it holds for any subquiver (extend the dimension vector by zero, and $B_{Q'}(\alpha) \geq B_Q(\bar{\alpha})$).

Lemma 4.2.1.
For extended ADE, has explicit β with $B(\beta) = 0$.



Thus finite representation type implies cannot contain extended Dynkin quiver.
Then it must be trivalent tree and of ADE type.

←) **ADE are finite representation type.**
Can check directly that $B(\alpha) \geq 1\ \forall \alpha \neq 0$ integral positive.

__`Weyl reflection' on ANY quiver__
to reduce indecomposable representation (positive root) to
simple representation (simple root) over vertices (of the new quiver).
(Underlying graph remains the same under reflection.)

Reflection functor $C_x: Rep(Q) \to Rep(Q')$ at a sink or source vertex x:
- Q': reverse the adjacent arrows of x.
- For rep. V:
  - Sink: replace the vector space at that vertex by sum of kernels of the adjacent arrows. (And the reversed arrows are the inclusions of

projections.)
Source: replace by cokernels.
For morphism $\phi: V \to V'$:
Reflected rep.: W, W'.
$\psi(x): W(x) \to W(x)'$: restriction of ϕ on kernels of adjacent arrows
(ϕ maps kernel to kernel').
(Or induced map on cokernels of adjacent arrows.
ϕ maps image to image')
For other vertices y, $\psi(y) = \phi(y)$.
Note: taking C_x does not lose any info. if surj. at sink x or inj. at source x.

Split away the simple S_x:
Lem. 4.3.6.
x sink: S_x is a direct summand of V iff the sum of adjacent arrows is not surjective.
x source: S_x is a direct summand of V iff the sum of adjacent arrows is not injective.

Proof.
Choose a complement of $\mathrm{Im} \subset V(x)$ and get $S_x^{\oplus k}$.
(or $S_x^{\oplus k}$ is Ker for source)
Replacing $V(x)$ by Im gives V'.
(or replace by complement of Ker.)
$V = V' \oplus S_x^{\oplus k}$.

Reflection "almost" sends indecomp. to indecomp.:
Thm. 4.3.9. [Bernstein-Gelfand-Ponomarev]
Have natural transformation
(x sink) $C_x^2 \to Id$
(x source) $Id \to C_x^2$.
1. If V indecomposable, $C_x(V)$ is indecomp. and $C_x^2(V) \cong V$, UNLESS $V \cong S_x$.
2. $V \cong S_x \Leftrightarrow C_x(V) = 0$.

In Case 1,
Lem. 4.3.6 $\Rightarrow$ adjacent arrows are surj. (for sink) or inj. (for source).
dim. vector becomes

$$\sigma_x(\alpha)\Big|_x = \sum_{y \text{ adj.to } x} \alpha(y) - \alpha(x)$$

and remains the same for other vertices.
In Case 2, for the same def. of σ, $\sigma_x(\alpha)|_x = -1$.
Hence can tell which case by considering $\sigma_x(\dim V)$.

Proof:
x sink:
$C_x^2(V)\Big|_x$ equals to $V(in_x)/Ker$, and equals to $V(y)$ on other vertices y.
Thus have natural morphism $C_x^2(V) \to V$
which is the induced map $V(in_x)/Ker \to V(x)$ at x, and Id on other vertices.
x source:
$C_x^2(V)\Big|_x$ equals to image of sum of adjacent arrows.
Thus have natural morphism $V \to C_x^2(V)$
which is the arrow map $V(x) \to Im$ at x.
Case 2 $V \cong S_x$ is easy computation (of kernel and cokernel).
Case 1 $V \ncong S_x$: Lem. 4.3.6 $\Rightarrow V(in_x)/Ker \overset{\cong}{\to} V(x)$ (surj) or $V(x) \overset{\cong}{\to} Im$ (inj).
If $C_x(V)$ is decomp., so is $V \cong C_x^2(V)$, a contradiction.

Strategy:
Given an indecomposable, keep on reflecting, until it get to zero. Then the representation right before zero is simple.
Key: need to do it systematically to ensure it get to zero.
Keep track by dimension vectors (which is positive before it gets to zero).
Then Coxeter element should send the dim. vector to neg. (get out of first quadrant)

<u>Weyl group</u>
σ_x: $\mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ with $\sigma_x^2 = Id$. (x does not need to be sink nor source.)
W: group generated by $\sigma_x, x \in Q_0$.
"Root": elements in a W-orbit of e_i.

Prop. 4.4.9. For ADE quiver,
1. Tits form B is pos. def. and preserved by W.

2. $B(\alpha) = 1$ for root α.
3. Only finitely many roots.
4. W is finite.
5. For root α, either $\alpha \geq 0$ or $\alpha \leq 0$.

Proof.
1 is by direct check.
2 follows from B being W-inv.
3 is by compactness of $\{B = 1\} \cap \mathbb{Z}^n$.
4 is $W \subset \mathrm{Perm}(\text{roots})$.
5: Write $\alpha = \alpha^+ - \alpha^-$ where $\alpha^+(x) = \alpha(x)$ if positive, $\alpha^-(x) = -\alpha(x)$ if neg.
$1 = B(\alpha) = B(\alpha^+) + B(\alpha^-) - 2\langle \alpha^+, \alpha^- \rangle \geq B(\alpha^+) + B(\alpha^-)$
since $\langle \alpha^+, \alpha^- \rangle = \sum_x \alpha^+(x)\alpha^-(x) - \sum_a \alpha^+(t_a)\alpha^-(h_a) = -\sum_a \alpha^+(t_a)\alpha^-(h_a) \leq 0$.
Then either $B(\alpha^+) = 0$ or $B(\alpha^-) = 0$ since B is pos. def. and integral.

Def. 4.4.10. Coxeter element:
$c = \sigma_{x_n} \dots \sigma_{x_1}$ product of all generators from base. (Choose an order.)

Lem. 4.4.12. Send to negative:
$c: \mathbb{R}^{Q_0} \to \mathbb{R}^{Q_0}$ has no fixed point other than 0.
Moreover, for any $\alpha \neq 0$, $c^k(\alpha)$ has neg. coord. for some $k \geq 0$.

<u>**Coxeter functor**</u>:
To realize for quiver, need to choose correct order so that vertices involved are sinks.
Def. 4.4.1.
Take a seq. of vertices $x_1, \dots, x_m$ such that x_{i+1} is sink in $Q_i = \sigma_{x_i} \dots \sigma_{x_1}(Q)$.
(In particular $x_i \neq x_{i+1}$.)
$C^+ := C_{x_m} \dots C_{x_1}: Rep(Q) \to Rep(Q')$;
$C^- := C_{x_1} \dots C_{x_m}: Rep(Q') \to Rep(Q)$. (Turn sources back to sinks.)

Lem. 4.4.3. Coxeter functor for trees.
Have such a sequence of sinks for trees with $m = |Q_0|$.
(Order it with $h_a < t_a$. $(1, \dots, n)$ gives such a sequence.)

Lem. 4.4.4.
Two quivers with the same underlying tree are related by $Q' = \sigma_{x_m} \dots \sigma_{x_1}(Q)$ for some sequence of sinks.

(Proof by induction.)

Lem. 4.4.2.
$Ind(Q)$: set of indecomposables.
$M := \left\{C_{x_1}^{\text{source}} \dots C_{x_{i-1}}^{\text{source}}\left(S_{x_i}^{\text{sink}} \in Ind(Q_{i-1})\right) \in Rep(Q): i = 1, \dots, m\right\}$.
$M' := \left\{C_{x_m}^{\text{sink}} \dots C_{x_{i+1}}^{\text{sink}}\left(S_{x_i}^{\text{source}} \in Ind(Q_i)\right) \in Rep(Q'): i = 1, \dots, m\right\}$.
$Ind(Q) - M \cong Ind(Q') - M'$ by $C^{\pm}$ (inverse to each other).
$M \xrightarrow{C^+} \{0\}, M' \xrightarrow{C^-} \{0\}$.

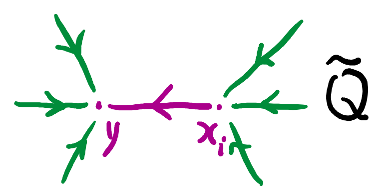
Immediately follows from Thm. 4.3.9.

Ex. 4.4.2.
Let $x_1 \dots x_m$ distinct.
$C_{x_1}^{\text{source}} \dots C_{x_{i-1}}^{\text{source}}\left(S_{x_i}^{\text{sink}} \in Ind(Q_{i-1})\right) = \mathbb{C}Q \cdot e_i$.
$C_{x_m}^{\text{sink}} \dots C_{x_{i+1}}^{\text{sink}}\left(S_{x_i}^{\text{source}} \in Ind(Q_i)\right) = (e_i \cdot \mathbb{C}Q')^*$.

Note that $S_{x_i}^{\text{sink}} = \mathbb{C}Q \cdot e_i$ since x_i is sink.

If reflect at adjacent source y, $C_y^{source}\left(S_{x_i}^{\text{sink}}\right) = \mathbb{C}\tilde{Q} \cdot e_i$ by def. (replacing 0 over y by cokernel which is $\mathbb{C}$).

Do induction.



Def. 4.4.7. Coxeter functor:
Given a sequence of sinks $x_1 \dots x_n$ where $n = |Q_0|$.
The corresponding $C^+, C^-: Rep(Q) \to Rep(Q)$ are called Coxeter functors.
(each arrow is adjacent to two vertices and hence reversed twice. Thus $Q' = Q$.)

By above exercises and Lem. 4.4.2,
M consists of all proj. indecomp. and M' consists of all inj. decomp.

Cor. 4.4.8.
V proj. $\Leftrightarrow C^+(V) = 0$.
V inj. $\Leftrightarrow C^-(V) = 0$.
V has no proj. summands $\Rightarrow C^-C^+(V) \cong V$.
V has no inj. summands $\Rightarrow C^+C^-(V) \cong V$.

(Think about indecomposables first)

Cor. 4.4.11.
If V has no proj. summand,
$C^+(V)$ is non-zero rep. with dim. $c(\alpha)$ (where $c = \sigma_{x_n} \dots \sigma_{x_1}$ and α is $\dim V$).

Lem. 4.4.6. Independent of order:
C^+ are naturally equivalent under (admissible) reordering of x_i.

Thm. 4.4.13.
If ADE, then $Ind(Q) \cong \{\text{pos. roots}\}$ by $V \mapsto \dim V$.
In particular finite.

Proof:
For $V \in Ind(Q)$,
$c^k(\dim V)$ is no longer positive for some $k > 0$.
Take min. such k.
Then ${C^+}^{k-1}(V) \neq 0$.
Also $\sigma_j \dots \sigma_1 c^{k-1}(\dim V)$ is no longer positive for some $j > 0$,
take min. such j.
Then $C_{j-1} \dots C_1 {C^+}^{k-1}(V) \neq 0$ and $C_j\left(C_{j-1} \dots C_1 {C^+}^{k-1}(V)\right) = 0$.
Hence $C_{j-1} \dots C_1 {C^+}^{k-1}(V) = S_j$.
$V = {C^-}^{k-1} C_1 \dots C_{j-1}(S_j)$.
$\dim V = c^{k-1} \sigma_1 \dots \sigma_j(e_j)$ which is a positive root by def.

Given pos. root α, do the same thing to get
$\sigma_{j-1} \dots \sigma_1 c^{k-1}(\alpha)$ positive but not $\sigma_j \dots \sigma_1 c^{k-1}(\alpha)$.
Then $\sigma_{j-1} \dots \sigma_1 c^{k-1}(\alpha) = \epsilon_j$.
(Recall that for root, coord. either all pos or all neg.)
Taking ${C^-}^{k-1} C_1 \dots C_{j-1}(S_j)$ gives an indecomp. rep.