Kac generalization of Gabriel's Theorem

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Ch. 8.

Main: for any quiver.
There exists indecomp. in α
$\Leftrightarrow$ α is a positive root of the corresponding root system (Kac-Moody Lie alg.)

Key:
Whether there exists indecomp. in α is indep. of orientation of arrows of Q.
Thus can change to sink/source to do reflection.

Deformed preprojective alg.

$$\Pi_\theta \coloneqq \mathbb{C}\bar{Q} \,/\, \left(\sum_{a \in Q_1} (a^* a - a a^*) \, - \theta \cdot Id \right).$$

where $\bar{Q}$ is the double quiver.
The relation is denoted as r_θ.
($\theta \in \mathbb{C}^{Q_0}$.)

Note:
r_θ consists of cycles (same source and target).

$Rep_\alpha(\Pi_\theta) = \mu^{-1}(\theta \cdot Id)$
where
$\mu\colon\ Rep_\alpha(Q) \oplus Rep_\alpha(Q^{op}) \to \mathrm{End}_\alpha(Q)$
is the "moment map"

$$\sum_{a \in Q_1} \big(W(a^*) V(a) - V(a) W(a^*) \big).$$

Moment map for linear action of $GL(n)$ on $M \oplus M^*$ (complex sympl.),
$(v, w) \mapsto (Av, wA^{-1})$
is
$M \oplus M^* \to \mathfrak{gl}^*$
$(\mu(v, w), X) = wXv.$
GL_α acts on $(V, W) \in Rep_\alpha(Q) \oplus Rep_\alpha(Q^{op})$ by
$\big(g_{h(a)} V(a) g_{t(a)}^{-1}, g_{t(a)} W(a^*) g_{h(a)}^{-1} \big).$

$\mathfrak{gl}_\alpha$ acts by
$\left(X_{h(a)}V(a) - V(a)X_{t(a)}, X_{t(a)}W(a^*) - W(a^*)X_{h(a)}\right)$.

$$(\mu(V,W),X) = \sum_a \operatorname{tr}\left(X_{h(a)}V(a)W(a^*) - V(a)X_{t(a)}W(a^*)\right)$$
$$= \sum_a \operatorname{tr}\left(X_{h(a)}V(a)W(a^*) - X_{t(a)}W(a^*)V(a)\right).$$

Image of
$$\mu = \sum_{a\in Q_1}\left(W(a^*)V(a) - V(a)W(a^*)\right)$$
has total trace zero. hence need
$\theta \cdot \alpha = \operatorname{tr}(\theta \cdot \mathrm{Id}) = 0$
or otherwise $\Pi_\theta = \emptyset$.

Recall the exact sequence
$$0 \to H^0 \to \bigoplus_x \operatorname{Hom}\left(V(x), W(x)\right) \to \bigoplus_a \operatorname{Hom}\left(V(t_a), W(h_a)\right) \to H^1 \to 0.$$
Put $V = W$:
$$0 \to \operatorname{End}(V) \to \operatorname{End}_\alpha(Q) \xrightarrow{d_V} \operatorname{Rep}_\alpha(Q) \to H^1(V) \to 0.$$
$(\alpha = \overrightarrow{\dim V}.)$
Dualizing: $(\sum_a Tr\left(W(a^*)V(a)\right)$ gives $\operatorname{Rep}_\alpha(Q^*) \cong \left(\operatorname{Rep}_\alpha(Q)\right)^*.)$
$$0 \to \left(H^1(V)\right)^* \to \operatorname{Rep}_\alpha(Q^*) \xrightarrow{d_V^*} \operatorname{End}_\alpha(Q^*) \to \left(\operatorname{End}(V)\right)^* \to 0$$
Indeed
$$d_V^* = \mu(V,-) = \sum_{a\in Q_1}\left((-)(a^*)\cdot V(a) - V(a)\cdot(-)(a^*)\right).$$

Thm. 8.1.3. Lift Q-rep V to Π_θ-rep.
$\theta \cdot Id \in \operatorname{Im}(d_V^*)$ (that is, V can be lifted)
$\Leftrightarrow \theta \cdot \overrightarrow{\dim W} = 0\ \ \forall$ indecomp. summand W of V.

Proof.
By exact seq,

$$\theta \cdot Id \in \operatorname{Im}(d_V^*) \Leftrightarrow \sum_x \theta(x) \cdot \operatorname{tr} f_x = 0 \ \forall f \in \operatorname{End}(V).$$

→)

consider $f = \pi_W \in End(V)$ projection to W.

$$\sum_x \operatorname{tr}(\theta(x)\, f_x) = \sum_x \theta(x) \operatorname{tr}(f_x) = \theta \cdot \overrightarrow{\dim W} = 0.$$

←)

Let $V = W_1 \oplus \cdots \oplus W_r$ indecomp.

$\theta\left(\overrightarrow{\dim W_i}\right) = 0 \ \forall i.$

Recall (from Ch.1) that

$\forall f \in \operatorname{End}(W_i)$,

f is spanned by Id and nilpotents (Jordan decomp. into gen. eigen-subrep.).

$$\sum_x \theta(x) \operatorname{tr}(f_x) = \sum_x \theta(x) \operatorname{tr}(\lambda \cdot Id) = \lambda\theta \cdot \overrightarrow{\dim W_i} = 0.$$

For $f \in End(V)$, write f into matrix form $f_{ij} \in \operatorname{Hom}(W_j, W_i)$.

$$\sum_x \operatorname{tr}(\theta(x)\, f_x) = \sum_x \theta(x) \sum_i \operatorname{tr}(f_{ii})_x = 0.$$

Cor. 8.1.4.

Suppose α is indivisible.

Whether there exists indecomp. in α is indep. of orientation of arrows of Q.

Proof.

Let Q, Q' be the same up to orientation.

Then they have the same double $\bar{Q}$.

Want:

Q has indecomp. in $\alpha \Leftrightarrow Q'$ has indecomp. in α.

Idea:

Lift Q-rep to Π_θ-rep, and then restrict to Q'-rep.

Suppose V indecomp. in α.

Choose $\theta \cdot \alpha = 0$.

Then by Thm. 8.1.3, V can be lifted to Π_θ-rep.

Restrict to Q'-rep. V' (with sam dim. α). Hope indecomp.

Again by Thm. 8.1.3,
$\theta \cdot \overrightarrow{\dim W} = 0 \ \ \forall$ indecomp. summand W of V'.

Claim:
can choose $\theta \in \langle\alpha\rangle_{\mathbb{C}}{}^{\perp}$ such that
$\theta \cdot \beta = 0$ and $\beta \in \mathbb{Q}^{Q_0} \Rightarrow \beta \in \mathbb{Q} \cdot \alpha$.
Since α indivisible, this implies $\overrightarrow{\dim W} \in \mathbb{Z}_{\geq 0} \cdot \alpha$.
Then the indecomp. W can only be V'.

Proof of claim:
Use irrational θ in $\langle\alpha\rangle_{\mathbb{C}}{}^{\perp} - \langle\alpha\rangle_{\mathbb{Q}}{}^{\perp}$.
Take basis $\gamma_1 \dots \gamma_{n-1}$ of $\langle\alpha\rangle_{\mathbb{Q}}{}^{\perp}$.
Take $\theta = \sum_{i=1}^{n-1} t_i \gamma_i$ where $t_i \in \mathbb{C}/\mathbb{Q}$ are lin. indep.
Then $\theta \cdot \beta = \sum_{i=1}^{n-1} t_i (\gamma_i \cdot \beta) = 0$
(where $\gamma_i \cdot \beta \in \mathbb{Q}$)
implies $\gamma_i \cdot \beta = 0 \ \forall i$, and hence $\beta \in \mathbb{Q} \cdot \alpha$.

Reflections for indecomp.

$\sigma_x(\alpha) := \alpha - (\alpha, \epsilon_x)\, \epsilon_x$
(Reflection about $\epsilon_x^{\perp}$ under the indef. "metric" $(-,-)/2$)
where

$$(\alpha, \beta) := \langle\alpha, \beta\rangle + \langle\beta, \alpha\rangle = 2 \sum_v \alpha(x)\beta(x) - \sum_a \alpha(t_a)\beta(h_a) - \sum_a \beta(t_a)\alpha(h_a).$$

$$\sigma_x(\alpha)\Big|_x = \alpha(x) - \left(2\alpha(x) - \sum_{\substack{v \text{ conn.to } x \\ \text{by arrow}}} \alpha(v)\right) = -\alpha(x) + \sum_{v \text{ adj to } x} \alpha(v).$$

Dually:
$\sigma_x^*(\theta) \cdot \alpha = \theta \cdot \sigma_x(\alpha)$.
$\sigma_x^*(\theta)\Big|_v = \theta(v) - (\epsilon_x, \epsilon_v)\, \theta(x)$.
$\sigma_x^*(\theta)\Big|_x = -\theta(x)$.

Weyl group
$W \coloneqq \langle \sigma_x : x \in Q_0 \rangle \subset GL(\mathbb{Z}^{Q_0})$.

Lem. 8.2.1. Generalizing Thm. 4.3.9 to non-sinks.
α indivisible.
If has indecomp. V in α, then
for every $x \in Q_0$,
has indecomp. rep. in $\sigma_x(\alpha)$
unless $\alpha = \epsilon_x$ in which case $V = S_x$
(and $\sigma_x(\epsilon_x) = -\epsilon_x$.)

Proof.
By Cor. 8.1.4,
can change the orientation such that x is sink,
and still has indecomp. V' (rep. of Q') in α.
Then use $C_x(V')$ in Thm. 4.3.9
which is still indecomp. (rep of $\sigma_x(Q')$) of dim. $\sigma_x(\alpha)$
unless $V = S_x$ (in which $C_x(S_x) = 0$).
Again by Cor. 8.1.4, has indecomp. rep. of the original Q in $\sigma_x(\alpha)$.

Note:
W preserves indivisibility: $\forall w \in W$,
α indivisible $\Leftrightarrow w \cdot \alpha$ indivisible.

Given an indecomp., keep on doing Lem. 8.2.1. When we encounter ϵ_x and want to do σ_x,
we replace ϵ_x by $-\epsilon_x$ and do σ_x (so that we get back ϵ_x). Thus we can do this for any $w \in W$.

Thm. 8.2.2.
Suppose $\alpha > 0$ indivisible.
Have indecomp. rep. of dim. α
$\Leftrightarrow$ have indecomp. rep. of dim. $\pm w(\alpha)$.
In particular either $w(\alpha) > 0$ or $-w(\alpha) > 0$ if α supp. indecomp.

<u>Reflections for Π_θ-mod</u>
$Z = (V, W) \in \Pi_\theta - \text{mod}$.
Fix $x \in Q_0$.
take $\theta(x) \neq 0$.

$$\sum_{a\in Q_1}\left(W(a^*)V(a)-V(a)W(a^*)\right)=\theta\cdot Id.$$

In particular
$Z(in_x)\cdot Z(out_x)=\theta(x)\cdot Id_{Z(x)}$
where
in_x is direct sum of inward arrows;
out_x is direct sum of outward arrows, corrected by signs:
multiply by -1 if the out arrow comes from Q_1^{op}.

Hence $\mathrm{Ker}(in_x)\cong\mathrm{coKer}(out_x)$;
$Z(in_x)$ is surj. and $Z(out_x)$ is inj.
(They give isom. $Z(x)\cong\mathrm{Im}\,Z(out_x)$.)

As in Ch.4,
$\sigma_x(Z)\Big|_x := \mathrm{Ker}(in_x)\cong\mathrm{coKer}(out_x)$ (where the isom. is the inclusion)
and remain the same for other vertices.
$\sigma_x(Z)(out_x)$ is the inclusion of kernel,
$\sigma_x(Z)(in_x)$ is the quotient map times $\lambda(x)$.
Remains the same for other arrows.

Lem. 8.2.4.
σ_x^2 is naturally equiv. to identity, and hence σ_x is an equiv.

Root system

In proving Gabriel's ADE case,
we used Coxeter element to get out of first quadrant,
and right before we get simple ϵ_x.

Another way to get to simple:
Suppose α supports indecomp. (and hence $\alpha>0$).
Consider (α,ϵ_x).
For ADE, $(\alpha,\alpha)>0$, and so
$(\alpha,\epsilon_x)>0$ for some x.
Then reflect at x.
$\sigma_x(\alpha)=\alpha-(\alpha,\epsilon_x)\,\epsilon_x$.

This decreases $\Sigma\alpha = \sum_y \alpha(y)$:
$\Sigma\sigma_x(\alpha) = \Sigma\alpha - (\alpha, \epsilon_x) < \Sigma\alpha$.
Keep on doing this,
until α is no longer > 0.
Right before still supports indecomp, so must be ϵ_x.

In gen. $(\alpha, \alpha) \ngtr 0$. So Weyl reflections results in two possibilities.

1. $w \cdot \alpha$ is no longer > 0.
   Right before the last σ_x, still supports indecomp, so must be ϵ_x.
   Such α is called **real root**.
   Note that such α must be indivisible.
2. $\alpha > 0$, but $(\alpha, \epsilon_x) \leq 0 \; \forall x$.
   Observe that α has indecomp. implies α has connected support,
   meaning there is a connected subquiver only containing
   those vertices with $\alpha(x) \neq 0$.
   Such α is called **imaginary root**.

This motivates the following definition of real and imaginary roots.

Real root:
$w \cdot \epsilon_x$ for any $w \in W, x \in Q_0$.
$\Phi_{re}^{+} = \Phi_{re} \cap \mathbb{Z}_{\geq 0}^{Q_0}$;
$\Phi_{re}^{-} = \Phi_{re} \cap \mathbb{Z}_{\leq 0}^{Q_0}$;

Since W preserves indivisibility,
all real roots are indivisible.

$\Phi_{re} = -\Phi_{re}$:
$-w \cdot \epsilon_x = w \cdot (-\epsilon_x) = (w \cdot \sigma_x) \cdot \epsilon_x \in \Phi_{re}$.

$\Phi_{re}^{-} = -\Phi_{re}^{+}$:
taking $-$ preserves Φ_{re} and switch $\mathbb{Z}_{\geq 0}^{Q_0}$ and $\mathbb{Z}_{\leq 0}^{Q_0}$.

(Positive or negative) imaginary root:
$\pm w \cdot \alpha$ for any $w \in W, \alpha > \vec{0}$,
$(\alpha, \epsilon_x) \leq 0 \; \forall x \in Q_0$, and
there is a connected subquiver only containing
those vertices with $\alpha(x) \neq 0$.

im

Note that if $\alpha \in \Phi_{im}$, then
$k\alpha \in \Phi_{im} \ \forall k \neq 0$.

Assume no loop at any vertex. Then
$\Phi_{re} \cap \Phi_{im} = \emptyset$:
$(\alpha, \alpha) = (\epsilon_x, \epsilon_x) = 2$ (since no loop) for $\alpha \in \Phi_{re}$,
but
$(\alpha, \alpha) = \sum_x \alpha(x) \cdot (\alpha, \epsilon_x) \leq 0$ for $\alpha \in \Phi_{im}$.

The above has proved $\rightarrow$ of the following:

Thm. 8.3.5.
For α indivisible,
α supports indecomp. rep. $\Leftrightarrow \alpha \in \Phi^+$.

$\leftarrow$ is proved below.

Lem. 8.3.1. Indecomp. for positive real root:
There exists (one) indecomp. rep. in every $\alpha \in \Phi^+_{re}$.

Proof.
By Thm. 8.2.2, have indecomp. rep. in $\pm w \cdot \epsilon_x$.
which has to lie in $\mathbb{Z}^{Q_0}_{\geq 0}$.

<u>Indecomp. rep. for positive imaginary root</u>

First show that if α satisfies the above condition, then
it supports indecomp. rep.
Then by Lem. 8.2.1,
restrict to the case that α indivisible,
still has indecomp. in $\sigma_x(\alpha) \ \forall x$.
(Note that $\sigma_x(\alpha) \neq \epsilon_x$ since $\Phi_{re} \cap \Phi_{im} = \emptyset$.)
Inductively get:

Lem. 8.3.4.
Every indivisible $\alpha \in \Phi^+_{im}$ supports indecomp.

First the case before taking $w \in W$:

Lem. 8.3.3.

If $\alpha > \vec{0}$, $(\alpha, \epsilon_x) \leq 0\ \forall x$, and
there is a connected subquiver only containing
those vertices with $\alpha(x) \neq 0$,
then has (inf. many) indecomp. rep. in α.

Proof.

Can shrink Q to be the connected full subquiver supporting α.
(Note that $(-,-)$ are the same for α supp. on the subquiver.)

Consider all possible decomp. rep., which lies in image of
$A \cdot (V \oplus W)\colon\ GL_\alpha \times Rep_\beta(Q) \times Rep_{\alpha-\beta}(Q) \to Rep_\alpha(Q)$
for $0 < \beta < \alpha$.
(The action of A is by conjugation at each vertex.)
Want: the image has positive codimension.

If A has diagonal block form, then image still lies in $Rep_\beta(Q) \oplus Rep_{\alpha-\beta}(Q)$.
Thus dim. of image is at most

$$\dim Rep_\beta(Q) + \dim Rep_{\alpha-\beta}(Q) + \bigoplus_{x \in Q_0} \left(\alpha(x)^2 - \beta(x)^2 - \left((\alpha-\beta)(x)\right)^2\right)$$

(the last term comes from anti-diagonal blocks of A).
Codim. is at least

$$\dim Rep_\alpha(Q) - \dim Rep_\beta(Q) - \dim Rep_{\alpha-\beta}(Q)$$
$$- \bigoplus_{x \in Q_0} \left(\alpha(x)^2 - \beta(x)^2 - \left((\alpha-\beta)(x)\right)^2\right).$$

Recall $\langle \alpha, \alpha \rangle = \dim \mathrm{End}_\alpha - \dim Rep_\alpha(Q)$.
Thus the above equals
$-\langle \alpha, \alpha \rangle + \langle \beta, \beta \rangle + \langle \alpha - \beta, \alpha - \beta \rangle = -(\beta, \alpha - \beta)$.
By the lemma below, this is ≥ 0, and equality holds iff
β is prop. to α and $\left(\alpha, \mathbb{Z}^{Q_0}\right) = 0$.

Done if > 0.

For the case β is prop. to α and $\left(\alpha, \mathbb{Z}^{Q_0}\right) = 0$ (where $\alpha(x) > 0\ \forall x$):

$\forall \gamma > \vec{0}$,

$(\gamma,\gamma) = (\gamma,\gamma - m\alpha) = -(\gamma, m\alpha - \gamma)$
where $m > 0$ is taken such that $\gamma < m\alpha$
(which exists since $\alpha(x) > 0\ \forall x$.)

Again by lemma below, RHS ≥ 0.
Hence $B(\gamma) \geq 0\ \forall \gamma \geq \vec{0}$.

See Ex. 4.2.5 that this implies Q is subgraph of extended Dynkin.
$B(\alpha) = 0 \Rightarrow Q$ cannot be Dynkin, and hence must be extended Dynkin.

By Lem. 4.2.2, α has to be multiple of the unique listed ones.
(Inf. many) indecomp. rep. in α can be explicitly written down.
(ex. think about the Jordan block for $\widetilde{A_0}$.)

Lem. 8.3.2.
If $\alpha \in \mathbb{Z}_{>0}^{Q_0}$ and $(\alpha, \epsilon_x) \leq 0\ \forall x$, then
$(\beta, \alpha - \beta) \leq 0\ \forall 0 < \beta < \alpha$,
$= 0 \Leftrightarrow \beta = c \cdot \alpha$ and $\left(\alpha, \mathbb{Z}^{Q_0}\right) = 0$.

Proof:
$2(\beta, \alpha - \beta) = \big((\alpha - \beta) + \beta, (\alpha - \beta) + \beta\big) - (\beta,\beta) - (\alpha - \beta, \alpha - \beta).\ \ (*)$

Similarly
$2\beta(x) \cdot (\alpha - \beta)(x) = \alpha(x)^2 - \beta(x)^2 - (\alpha - \beta)(x)^2$.

Recall
$$(\alpha,\beta) = 2\sum_v \alpha(v)\beta(v) - \sum_a \alpha(t_a)\beta(h_a) - \sum_a \beta(t_a)\alpha(h_a).$$
In particular $\left(\epsilon_x, \epsilon_y\right) \leq 0$ for $x \neq y$.
(But (ϵ_x, ϵ_x) can be arbitrary.)

To make use of $(\alpha, \epsilon_x) \leq 0\ \forall x$, write the first term of $(*)$:
$$(\alpha,\alpha) = \sum_x \alpha(x) \cdot (\alpha, \epsilon_x).$$
Using the above equality,
$$\alpha(x) = \frac{2\beta(x)}{\alpha(x)} \cdot (\alpha - \beta)(x) + \frac{\beta(x)^2}{\alpha(x)} + \frac{(\alpha - \beta)(x)^2}{\alpha(x)} \geq \frac{\beta(x)^2}{\alpha(x)} + \frac{(\alpha - \beta)(x)^2}{\alpha(x)}.$$
Thus

$$(\alpha,\alpha) \leq \sum_x \frac{\beta(x)^2}{\alpha(x)}(\alpha,\epsilon_x) + \frac{(\alpha-\beta)(x)^2}{\alpha(x)}(\alpha,\epsilon_x).$$

Consider the first term:

$$\sum_x \frac{\beta(x)^2}{\alpha(x)} \cdot (\alpha,\epsilon_x) - (\beta,\beta)$$

$$= \sum_{x,y} \left(\frac{\beta(x)^2\alpha(y)}{\alpha(x)} - \beta(x)\beta(y) \right) (\epsilon_x,\epsilon_y)$$

$$= \sum_{x,y} \alpha(x)\alpha(y) \left(\left(\frac{\beta(x)}{\alpha(x)}\right)^2 - \frac{\beta(x)\beta(y)}{\alpha(x)\alpha(y)} \right) (\epsilon_x,\epsilon_y)$$

$$= \sum_{x,y} \frac{\alpha(x)\alpha(y)}{2} \left(\frac{\beta(x)}{\alpha(x)} - \frac{\beta(y)}{\alpha(y)} \right)^2 (\epsilon_x,\epsilon_y) \leq 0.$$

Similarly replacing β by $\alpha-\beta$, we get the comparison for the second term.
Get $(\beta,\alpha-\beta) \leq 0$.

Equality holds if and only if
$\frac{\beta(x)}{\alpha(x)} = \frac{\beta(y)}{\alpha(y)}$ $\forall x, y$ and $\frac{2\beta(x)}{\alpha(x)} \cdot (\alpha-\beta)(x)\,(\alpha,\epsilon_x) = 0$.
The first cond. gives $\beta = c\alpha$ $(\beta \neq 0)$.
The second cond. implies $(\alpha,\epsilon_x) = 0$ since $\beta \neq \alpha$.

<u>Quiver over other fields</u>
Can remove indivisible condition by using finite fields.

Reduction mod p:
Prop. 8.4.8.
Has k (or inf.) indecomp. in α over $\mathbb{C}$
iff has k (or inf.) such over $\overline{\mathbb{F}_p}$ for inf. many prime.

$\overline{\mathbb{F}_p}$ is union of $\mathbb{F}_q$, $q = p^k$ for prime p.

Advantage of $\mathbb{F}_q$, $q = p^k$ for prime p:
Finite number of rep. classes in α.

indecomp. in $\overline{\mathbb{F}_p}$ is $(k \to \infty)$-limit of
absolutely indecomp. in $\mathbb{F}_{p^k}$ defined below.

For field extension F' over F, have "pull-back"
$F' \otimes_F V$ (rep. over F')
for rep. V over F.
(Take $1 \otimes_F V(a)$ for arrows a.)

Note:
can still regard $F' \otimes_F V$ as rep. over F, which is isom. to
$V^{\oplus d}$
where d is deg. of field ext.

Absolutely indecomp.:
$\bar{F} \otimes_F V$ is indecomp.

"Pulling back" is injective:
Lem. 8.4.1.
$F' \otimes_F V \cong F' \otimes_F W$ over $F' \Leftrightarrow V \cong W$ over F.
Proof.
$\leftarrow$ is clear.
$\Rightarrow$)
Isom. over F' is also isom. over F.
$V^{\oplus d} \cong W^{\oplus d}$
and hence $V \cong W$
by Krull-Remak-Schmidt theorem (which holds over any field).
(Same indecomp. summands and same multiplicities.)

Strategy:
$a_\alpha := \#(\text{abs. indecomp. in } \alpha)$
is indep. of arrow orientation.
(Don't need indivisibility of α this time.)
Also $a_{\sigma_x \cdot \alpha} = a_\alpha$
if $a_\alpha \neq 0$ and $\alpha \neq \epsilon_x$.
Then reduce to either ϵ_x (real root)
or (im. root)
those $\alpha > 0$, but $(\alpha, \epsilon_x) \leq 0$ and α has conn. supp.
in which case we already know
$a_\alpha = 1$ or ∞ respectively.

Work over F_q now, $q = p^k$ for prime p.
First show that
  # rep. in α is indep. of orientation of Q.
Then show that
  # indecomp. in α can be written in terms of
  # rep. in β for all $\beta \leq \alpha$,
  and hence also indep. of orientation of Q.
Finally show that
  # abs. indecomp. in α can be expressed in term of
  # indecomp. in α/k for all $k|\alpha$,
  and hence also indep. of orientation of Q.

Lem. 8.4.2.
$|V/G| = |V^*/G|$
for finite dim. v.s. V.

Lem. 8.4.3.
$|(V \oplus W)/G| = |(V^* \oplus W)/G|$.

Lem. 8.4.4.
Number of rep. in α does not depend on orientation.

Proof.
Count number of orbits of $GL_\alpha(F_q)$ in $Rep_\alpha(Q; F_q)$.
Reversing arrow: taking $\left(Hom(V_{h_a}, V_{t_a})\right)^* \cong Hom(V_{t_a}, V_{h_a})$.
Same number of orbits by Lem. 8.4.4.

Lem. 8.4.5.
i_α, the number of indecomp. rep. in α, does not depend on orientation.

Proof.
Take generating function.
$$\prod_{\alpha>0} \frac{1}{(1-t^\alpha)^{i_\alpha}} = \prod_{\alpha>0} (1 + t^\alpha + t^{2\alpha} + \cdots) \dots (1 + t^\alpha + t^{2\alpha} + \cdots)$$
where i_α is the number of distinct indecomp. in α.
Each term is: product of $t^{k\alpha}$ from each factor, representing k copies of certain class

of indecomp. of dim. α.
Hence it equals to the generating function of rep.

$$\sum_{\alpha\geq 0} r_\alpha\, t^\alpha$$

where r_α is number of distinct rep. in α.
Thus i is determined by r.
Then follows from Lem. 8.4.4.

Now we need to study relation between
indecomp. and abs. indecomp.
Given indecomp. V over F_q, take
$\overline{F_q} \otimes_{F_q} V$ over $\overline{F_q}$.
Let W be an indecomp. summand.
Then W is defined over F_{q^d} for some d.
So $F_{q^d} \otimes_{F_q} V$ has W as indecomp. summand,
and $F_{q^d} \otimes_{F_q} V$ is abs. indecomp.
This means,
any indecomp. after pulled back to F_{q^d} for some d
has an abs. indecomp. summand.

The following lemma shows that
$F_{q^d} \otimes_{F_q} V = W \oplus \Phi(W) \oplus \cdots \oplus \Phi^{e-1}(W)$
where Φ is the **Frob. aut.**
$\Phi: F_{q^d} \to F_{q^d}, x \mapsto x^q$. ($q = p^k$.)
Fixed set is the subfield F_q in F_{q^d}.

Thus the indecomp. V over F_q
gives e abs. indecomp. over F_{q^d}
(if we take d as above).

Lem. 8.4.6.
V: indecomp. over F_q.
Take $F_{q^d} \otimes_{F_q} V$.
Let W be indecomp. summand (over F_{q^d}).
Then

$F_{q^d} \otimes_{F_q} V = W \oplus \Phi(W) \oplus \cdots \oplus \Phi^{e-1}(W)$
(Φ is Frob. aut. on every matrix entry)
where e is smallest such that $W \cong \Phi^e(W)$.

Proof.
For rep. W over F_{q^d},
take $F_{q^d} \otimes_{F_q} W$ (where W is taken over F_q here).
$F_{q^d} \otimes_{F_q} W \cong W \oplus \Phi(W) \oplus \cdots \oplus \Phi^{d-1}(W)$:
Vertices:
$\lambda \otimes v \mapsto \lambda\left(v, \Phi(v), \ldots, \Phi^{d-1}(v)\right)$.
Injective and same dim., and hence iso.
Arrows:
$1 \otimes W(a)$ is identified with
$\mathrm{Diag}\left(W(a), \Phi\big(W(a)\big), \ldots, \Phi^{d-1}\big(W(a)\big)\right)$.

Note that
$\Phi\left(F_{q^d} \otimes_{F_q} V\right) = F_{q^d} \otimes_{F_q} V$ since arrow is $1 \otimes V(a)$ defined over F_q.
Thus once W is a direct summand, $\Phi^k(W)$ is also a direct summand.
$F_{q^d} \otimes_{F_q} W$ is then a direct summand of $\left(F_{q^d} \otimes_{F_q} V\right)^{\oplus k}$.

Regard these as over F_q for the moment.
Since V indecomp., the above implies W (over F_q) is $V^{\oplus l}$.
Then $F_{q^d} \otimes_{F_q} W = \left(F_{q^d} \otimes_{F_q} V\right)^{\oplus l}$.

$F_{q^d} \otimes_{F_q} W$
$\cong W \oplus \Phi(W) \oplus \cdots \oplus \Phi^{d-1}(W) = \big(W \oplus \Phi(W) \oplus \cdots \oplus \Phi^{e-1}(W)\big)^{\oplus \frac{d}{e}}$.
All $\Phi^i(W)$ are indecomp. Thus
$F_{q^d} \otimes_{F_q} V = \big(W \oplus \Phi(W) \oplus \cdots \oplus \Phi^{e-1}(W)\big)^m$.

$m = 1$:
V is indecomp.
Consider $\mathrm{End}\left(F_{q^d} \otimes_{F_q} V\right) \cong F_{q^d} \otimes_{F_q} \mathrm{End}(V)$.
Every element in $\mathrm{End}(V)$ is either invertible or nilpotent.

Thus every non-zero element in $F_{q^d} \otimes_{F_q} \mathrm{End}(V)/\mathfrak{n}$ is invertible.
But this is not true for $\mathrm{End}(M^m)$ for $m \geq 2$.
($\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is neither invertible nor nilpotent.)

Then we get a (better) version of Thm. 8.2.2 over $\mathbb{F}_q$.

Lem. 8.4.7.

a_α, the number of abs. indecomp. in α over $\mathbb{F}_q$, does not depend on orientation of arrows.
(Now don't need indivisibility.)

Proof.

Argue that a_α can be written in terms of i_α,
which is indep. of arrow orientation.
Given indecomp V, by Lem. 8.4.6,
$F_{q^k} \otimes_{F_q} V = W \oplus \Phi(W) \oplus \cdots \oplus \Phi^{e-1}(W)$
and W is abs. indecomp. for k big enough.
Take min. such k. Then $e = k$.
Thus

$$i_\alpha = \sum_{k|\alpha} b_{k,\alpha/k}(q)/k$$

where $b_{k,\alpha/k}$ is #abs. indecomp. over F_{q^k}
that does not comes from previous F_{q^l}.
Then
$b_{k,\alpha/k}(q) = a_{\alpha/k}(q^k) - \cdots$
where the later terms involve $a_{\alpha/k}(q^l)$ for $l|k$.
In particular $b_{1,\alpha}(q) = a_\alpha(q)$.
Thus

$$i_\alpha = \sum_{k|\alpha} b_{k,\alpha/k}(q)/k = a_\alpha(q) + \cdots$$

where the later terms only involves $a_{\alpha/k}$,
which can be written in terms of $i_{\alpha/k}$ by induction.

Thm. 8.4.10.

For $\alpha > 0$ that supports abs. indecomp.,
$a_\alpha = a_{\pm w(\alpha)}$

where we take the one in $\pm w(\alpha)$ that is > 0.

Proof.
By Lem. 8.4.7, can make a vertex into sink,
and then do reflection at that vertex.
Unless $\alpha = \epsilon_x$, σ_x gives one-one for abs. indecomp.
For $\alpha = \epsilon_x$, take $-\sigma_x(\epsilon_x) = \epsilon_x$ and do nothing.
This realizes $\pm w$.

Thm. 8.4.11.
Now over $\mathbb{C}$.
α is pos. real root iff it has exactly one indecomp. (assuming no self loop)
pos. im. root iff it has inf. many indecomp.

Proof.
First consider $\overline{F_p}$.
By Thm. 8.4.10,
reduce to either ϵ_x (real case) or
those $\alpha > 0$, but $(\alpha, \epsilon_x) \leq 0$ and α has conn. supp. (im. case)
Already known they have one (real) (assuming no self loop)
or inf. many (im.) indecomp.
Then by Prop. 8.4.8. also true for $\mathbb{C}$.