GIT

Monday, December 30, 2019 9:47 AM

Section 9 and [King]

<u>Affine group action</u>

G: aff. alg. gp.

(ex. GL. $\{A: \det A \neq 0\} = \{(A, y): y \det A = 1\}$.)

X: aff. var.

G acts on X.

Action $G \times X \to X$ gives homo.

$\mu^*: \mathbb{C}[X] \to \mathbb{C}[G] \otimes \mathbb{C}[X]$.

Let $\mu^*(f) = \sum h_i \otimes f_i$. Then by def.

$$(g \cdot f) = \sum h_i(g^{-1})\, f_i \,.$$

Lem. 9.1.13.

$W \subset \mathbb{C}[X]$ f.d. subspace.

$G \cdot W \coloneqq \mathrm{Span}\{g \cdot w\}$ is f.d. G-rep.

Proof.

$\mu^*(W)$ is f.d. and hence contained in some f.d. $A \otimes B$.

Then $G \cdot W \subset B$ and hence f.d.

Prop. 9.1.15: Put X in a rep.

Have rep. V and G-equiv. closed immersion (iso. to its image which is a subvar.)

$\phi: X \to V$.

Proof.

Key: take $W \subset \mathbb{C}[X]$ f.d. subspace that gen. $\mathbb{C}[X]$.

$V \coloneqq (G \cdot W)^*$.

Then $\mathbb{C}[V] = \mathbb{C}[z_1, \dots, z_n]$

where $z_1, \dots, z_n$ is a basis of $G \cdot W \subset \mathbb{C}[X]$.

Have $\mathbb{C}[V] = \mathbb{C}[z_1, \dots, z_n] \to \mathbb{C}[X]$

which is surj. since $G \cdot W$ gen. $\mathbb{C}[X]$.

This gives $X \to V$ which is G-equiv.

Apply to $X = G$, get G-equiv. $\phi: G \subset V$.

$G \to GL(V)$ is injective. (If acts as Id on V, then Id on G.)
Thus aff. alg. group must be lin.

<u>**Alg. framework for integration**</u>

Def. 9.2.1. Reynolds operator:

$R: \mathbb{C}[G] \xrightarrow{\text{lin}} \mathbb{C}$ with
$R(1) = 1;$
$R(h \cdot f) = R(f)\ \forall h, f.$
Linear reductive if such R exists.

(Recall `reductive' means complexification of compact Lie group.)

Main example: $GL(n, \mathbb{C})$.
$R(f) := \int_{U(n)} f\Big|_{U(n)}\, d\mu\,.$
$R(h \cdot f) = R(f)$ for $h \in U(n)$.
$U(n) \subset GL(n, \mathbb{C})$ is Zariski dense:
- $(\mathbb{S}^1)^n \subset (\mathbb{C}^\times)^n$ is Zariski dense.
- (Restrict to one variable, polynomial only have finitely many roots.)
- Have singular value decomp. $GL = U_1 D U_2^{-1}$.
- Thus $U(n) = U(n) \cdot (\mathbb{S}^1)^n \cdot U(n)$ is Zariski dense.

$R(h \cdot f) - R(f)$ is polynomial on $h \in GL(n, \mathbb{C})$. Thus
$R(h \cdot f) = R(f)$ for $h \in U(n)$ implies for $h \in GL(n, \mathbb{C})$.

Averaging for $f \in \mathbb{C}[X]$ where G acts on X: Analog of
$\bar{f}(x) := \int_K f(h \cdot x)\, d\mu.$
(f is poly. implies invariance under K gives invariance under G.)
$R_X: \mathbb{C}[X] \to \mathbb{C}[X]^G$ which is composition of
$\mathbb{C}[X] \xrightarrow{\mu^*} \mathbb{C}[G] \otimes \mathbb{C}[X] \xrightarrow{R \otimes 1} \mathbb{C}[X].$

The following is direct verification:
Lem. 9.2.4.
1. $R_X(\mathbb{C}[X]) \subset \mathbb{C}[X]^G$.
2. $R_X(f) = f$ for $f \in \mathbb{C}[X]^G$.
3. R_X is $\mathbb{C}[X]^G$-mod. homo.

4. For $W \subset \mathbb{C}[X]$ G inv., $R_X(W) = W^G$.

Aff. reductive implies semi-simple

(A more alg. way avoiding inv. metric)

Lem. 9.2.9.

Restriction of pairing to $V^G \times (V^*)^G \to \mathbb{C}$ is non-deg.

(Note that $(V^*)^G$ is different from $\left(V^G\right)^*$.)

Proof.

For $v \neq 0 \in V^G \subset V$, take $f \in V^*$ with $f(v) = 1$.

$R_V(f) \in (V^*)^G$ and $R_V(f)(v) = 1$ since v is G-inv.

(Van. ideal of $\{v\}$ is G-inv.)

Prop. 9.2.11.

Every rep. of aff. reductive group is a direct sum of irred.

Proof.

Let $W \subset V$ irred. subrep.

Natural pairing

$\mathrm{Hom}(V, W) \times \mathrm{Hom}(W, V) \to \mathbb{C}$

given by $tr_W(\psi\phi)$. Restrict to non-deg.

$\mathrm{Hom}(V, W)^G \times \mathrm{Hom}(W, V)^G \to \mathbb{C}$

by Lem. 9.2.9.

Let $\iota \in \mathrm{Hom}(W, V)^G$ be the inclusion.

There is $\psi \in \mathrm{Hom}(V, W)^G$ such that $tr(\psi\iota) \neq 0$.

$0 \neq \psi\iota \in \mathrm{Hom}(W, W)^G = \mathbb{C} \cdot Id$ (Schurs).

Then $V = W \oplus \mathrm{Ker}(\psi)$.

Quotient is finitely generated

First consider G-rep. $X = V$.

$V//G := \mathrm{MaxSpec}\left(\mathbb{C}[V]^G\right)$ with Zar. top.

$f \in \mathbb{C}[V]^G$ is called an "invariant".

Thm. 9.2.6. Hilbert's Finiteness Theorem.

$\mathbb{C}[V]^G$ is finitely generated as alg.

Proof.
Since G acts linearly,
$\mathbb{C}[V]_d$ is preserved by G, and hence $R_V(\mathbb{C}[V]_d) = \mathbb{C}[V]_d{}^G$.
Since $\mathbb{C}[V]$ is Noetherian, any ideal is fin. gen.
Take $\mathfrak{m} \coloneqq \bigoplus_d \mathbb{C}[V]_d{}^G \subset \mathbb{C}[V]^G$ max ideal.
$I \coloneqq \mathbb{C}[V] \cdot \mathfrak{m} \subset \mathbb{C}[V]$ ideal.
Let $f_1 \dots f_r \in \mathfrak{m}$ homog. gen. of I.
$f_1 \dots f_r$ gen. ideal $\mathfrak{m} \subset \mathbb{C}[V]^G$:
Any $h = \sum a_i f_i$ for $a_i \in \mathbb{C}[V]$.

$$h = R_V(h) = \sum R_V(a_i)\, f_i$$

where $R_V(a_i) \in \mathbb{C}[V]^G$.
By the following Prop. 9.2.5,
$f_1 \dots f_r$ gen. $\mathbb{C}[V]^G$.

Cor. 9.2.8.
For aff. X acted by G, $\mathbb{C}[X]^G$ is finitely generated.

Proof.
Put $X \subset V$ equivariantly by Prop. 9.1.15. Thus
$\mathbb{C}[X]^G = \iota^*\left(\mathbb{C}[V]^G\right)$ is fin. gen.

Prop. 9.2.5.
R graded. comm. alg.

$$\mathfrak{m} = \bigoplus_{d \geq 1} R_d\,.$$

If homog. $f_1 \dots f_r \in \mathfrak{m}$ gen. $\mathfrak{m}$ as ideal, then
$f_1 \dots f_r$ gen. R. as alg. (meaning only taking sums and products of f_i.)

Proof.
Any $h \in R_d$ can be generated:
Induction on d.

$$h = \sum a_i f_i \text{ for } a_i \in R.$$

f_i homog. implies can take a_i homog. with lower deg.

a_i gen. by f_i by induction.

GIT quotient for affine case with trivial character:

$X//G := \mathrm{MaxSpec}\big(\mathbb{C}[X]^G\big)$.

Have $\pi: X \to X//G$ from $\mathbb{C}[X]^G \to \mathbb{C}[X]$.

Main:

$\pi^{-1}(\{\pi(x)\})$ is a union of the orbits whose closure intersect with $\overline{G \cdot x}$.

Lem. 9.3.1.

For closed inv. $A_1, A_2 \subset X$,

$\overline{\pi(A_1)} \cap \overline{\pi(A_2)} = \overline{\pi(A_1 \cap A_2)}$.

Proof.

$\overline{\pi(A_i)}$ corr. to $J_i \cap \mathbb{C}[X]^G = R_X(J_i)$ where J_i is van. ideal of A_i.

$\overline{\pi(A_1)} \cap \overline{\pi(A_2)}$ corr. to

$R_X(J_1) + R_X(J_2) = R_X(J_1 + J_2) = (J_1 + J_2) \cap \mathbb{C}[X]^G$

which corr. to $\overline{\pi(A_1 \cap A_2)}$.

Lem. 9.3.2.

$\pi: X \to X//G$ is surj.

For closed inv. $A \subset X$, $\pi(A)$ is closed. Thus Lem. 9.3.1 simplifies to

$\pi(A_1) \cap \pi(A_2) = \pi(A_1 \cap A_2)$ for closed inv. $A_1, A_2 \subset X$.

Proof.

Surjective:

For max. ideal $I \subset \mathbb{C}[X]^G$, take ideal $J = \mathbb{C}[X] \cdot I \subset \mathbb{C}[X]$.

$J \cap \mathbb{C}[X]^G = I$:

For $h = \sum a_i f_i$ in LHS,

$$h = R_X(h) = \sum R_X(a_i)\ f_i \in I.$$

J is contained in some max. ideal $\mathfrak{m}$, and $\mathfrak{m} \cap \mathbb{C}[X]^G = I$.

$\pi(A)$ is closed:

Supposed not closed. Have $y \in \overline{\pi(A)} - \pi(A)$.

$B := \pi^{-1}(y) \neq \emptyset$ is closed inv.

$y \in \overline{\pi(A)} \cap \overline{\pi(B)} = \overline{\pi(A \cap B)} = \emptyset$!

Prop.
$G \cdot x$ and $G \cdot x'$ sit in the same fiber of π iff $\overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset$.

Proof.
First, fiber of π is closed, and hence contains a whole orbit closure.
$\pi(\overline{G \cdot x}) \cap \pi(\overline{G \cdot x'}) = \pi(\overline{G \cdot x} \cap \overline{G \cdot x'})$.
Thus whether the two image points are the same are determined by whether $\overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset$ or not.

Cor. 9.3.3.
$\pi^{-1}(y)$ contains a unique closed orbit.

Proof.
Exist:
For $z \in \pi^{-1}(y)$, if $\overline{G \cdot z} \neq G \cdot z$, take
$z_1 \in \overline{G \cdot z} - G \cdot z \subset \pi^{-1}(y)$.
Keep on doing this, get
$z_1, z_2 \dots$
and
$\overline{G \cdot z_1} \subset \overline{G \cdot z}, \qquad \overline{G \cdot z_2} \subset \overline{G \cdot z_1} \dots$
Since Noetherian, gradually stabilizes and
$\overline{G \cdot z_k} = G \cdot z_k$.
Unique:
Suppose have two distinct closed $G \cdot z_i$ for $i = 1,2$.
By Lem. 9.3.2,
$y \in \pi(G \cdot z_1) \cap \pi(G \cdot z_2) = \pi(G \cdot z_1 \cap G \cdot z_2) = \emptyset$!

For linear $V//G$, **Hilbert nullcone**:
$N := \pi^{-1}\{\pi(0)\}$.
$N = \{v\colon \overline{G \cdot v} \ni 0\}$: immediate from the above prop.

GIT quotient for general case
General X quasi-proj:
L: equivariant ample line bundle over X.

$R = \bigoplus_{n\geq 0} \Gamma(X, L^{\otimes n})$.

$X^{ss} := X - \mathrm{Zero}(R_+^G) = \cup_s \ U_s$

where $U_s = \{s \neq 0\}$ for $s \in \Gamma(X, L^{\otimes n>0})^G$.

(ss stands for "semi-stable".)

$X//_L G$ is glued from $U_s//_L G = \mathrm{Spec}(A_s^G)$ where $U_s = \mathrm{Spec}(A_s)$.

When $X = \mathrm{Proj}\ R$,

$X//G = \mathrm{Proj}\ R^G$.

$\pi\colon X^{ss} \to X//G$

is the glued version of the affine case before. Still have Lem. 9.3.2:

$A \subset X//G$ closed iff $A \subset X^{ss}$ closed.

$X^s := \{x \in X^{ss}\colon G \cdot x$ is closed in X^{ss} and finite stabilizer$\}$ is open.

$\pi|_{X^s}$ coincides with set-theoretic quotient.

ex. $X = \mathbb{P}(V)$ with trivial char.:

$L = O(1)$ with trivial action.

$R = \mathbb{C}[V]$;

$\mathrm{Zero}(R_+^G) = N$;

$V^{ss} = V - N$;

$X^{ss} = \mathbb{P}(V^{ss})$;

$X//G = \mathrm{Proj}\ R^G = \mathbb{P}(V//G)$.

$\pi\colon X^{ss} \to X//G$

descended from $V^{ss} \to V//G$.

GIT quotient for linear action with character:

(Section 2 of [King])

$f \in \mathbb{C}[V]$, $\chi\colon G \xrightarrow{\text{homo}} \mathbb{C}^\times$ such that

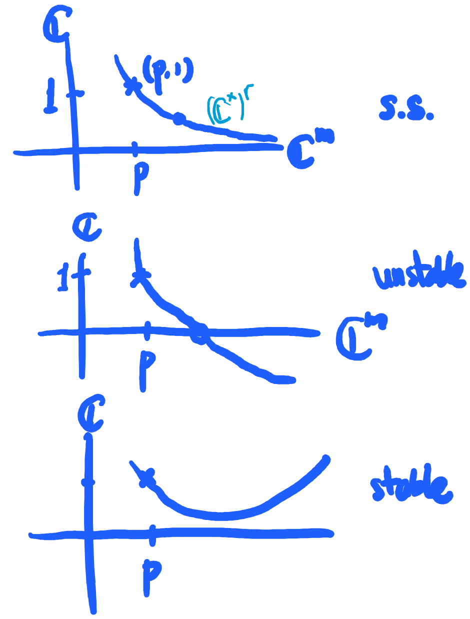
$g \cdot f = \chi(g)\, f \ \ \forall g$.

f is called a semi-invariant of weight χ.

χ can be understood as G-equiv. trivial line bundle

$L^{-1} = V \times \mathbb{C}, \qquad g \cdot (x, z) = \big(g \cdot x, z \cdot \chi^{-1}(g)\big)$.

deg $= n$ invariant section (thought as $L^{\otimes n}$):
$f(x)z^n \in \mathbb{C}[V \times \mathbb{C}]$ where $f(g \cdot x) = \chi^n(g)\, f(x)$.



Def.

$$V//_\chi G \coloneqq \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[V]^{G,\chi^n}\right)$$

which is proj. over $V//G = \mathrm{Spec}\big(\mathbb{C}[V]^G\big)$.

Geom. description:
$V//_\chi G = V^{\chi-ss}/\!\sim$ where
$x \sim x'$ iff $\overline{G\cdot x} \cap \overline{G \cdot x'} \neq \emptyset$
(where the closure is taken in $V^{\chi-ss}$)
iff $\overline{G\cdot(x,1)} \cap \overline{G\cdot(x',1)} \neq \emptyset$
(where closure is taken in $V \times \mathbb{C}$)

Semi-stable $x \in V$: $\exists f \in \mathbb{C}[X]^{G,\chi^n}$ for $n \geq 1$ such that $f(x) \neq 0$.
(Orbit closure of $(x,1) \in V \times \mathbb{C}$ is disjoint from zero-section.)

Stable: furthermore, $\mathrm{Stab}_{(x,1)}/Ker$ is finite (iff $\dim G\cdot x = \dim G/Ker$)
and G-action on $\{u \in V: f(u) \neq 0\}$ for the f above has closed orbits.
($G \cdot (x,1)$ is closed in $V \times \mathbb{C}$ iff $G \cdot x$ is closed in $V^{\chi-ss}$)
(Ker is the kernel of the linear G-action on V. Assume $\chi(\mathrm{Ker}) = 1$.)

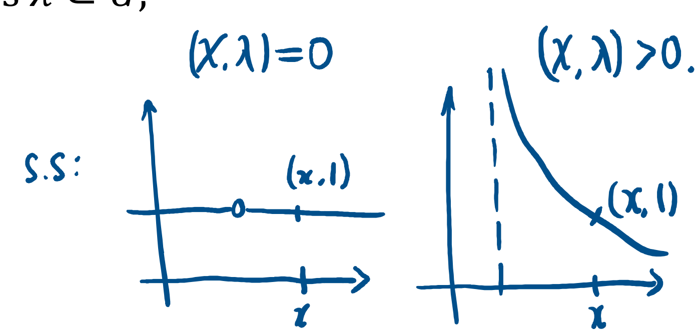
Reason for geom. description:
disjoint closed G-sets can be distinguished by G-inv. functions.

Hilbert-Mumford numerical criterion:
x is **s.s.** iff for all one-parameter subgroups $\lambda \subset G$,

$$\lim_{t\to 0} \lambda(t)\cdot(x,1) \notin V \times \{0\}$$

iff

$$\lim_{t\to 0} \lambda(t)\cdot x \text{ exists} \Rightarrow (\chi,\lambda) \geq 0.$$



stable iff

$$\lim_{t\to 0} \lambda(t)\cdot(x,1) \text{ exists} \Rightarrow \lambda \subset \Delta$$

iff

iff

$\lim_{t\to 0} \lambda(t)\cdot x$ exists $\Rightarrow (\chi,\lambda) > 0$ or $\lambda \subset \Delta$.

(χ,λ) is defined by

$\chi\big(\lambda(t)\big) = t^{(\chi,\lambda)}: \mathbb{C}^\times \to \mathbb{C}^\times$. $\lambda(t)\cdot(x,1) = \left(\lambda(t)\cdot x, \chi\big(\lambda(t)\big)^{-1}\right)$.

Note: for x s.s., $G\cdot(x,1)$ is closed iff

$\lim_{t\to 0} \lambda(t)\cdot x$ exists and $(\chi,\lambda) = 0 \Rightarrow \lim_{t\to 0} \lambda(t)\cdot x \in G\cdot x$.

In such case $\lim_{t\to 0}\lambda(t)\cdot x$ is fixed by λ.

If further x has finite stabilizer, so does $\lim_{t\to 0}\lambda(t)\cdot x$. Then $\lambda \subset \Delta$.

For $x, y \in V^{\chi-ss}$, $\boldsymbol{x \sim y}$ iff

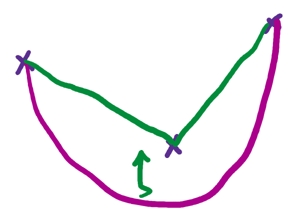
$\exists \lambda_1, \lambda_2$ with $\chi(\lambda_1) = \chi(\lambda_2) = 0$ such that

$\lim_{t\to 0}\lambda_1(t)\cdot x$ and $\lim_{t\to 0}\lambda_2(t)\cdot y$ belong to the same closed G-orbit.

Principle:

If a closed G-set A intersects $\overline{G\cdot x}$, then A intersects $\overline{\lambda\cdot x}$ for some one parameter λ.

For the last statement for $x\sim y$, $\overline{G\cdot(x,1)} \cap \overline{G\cdot(y,1)}$ is a closed G-set which must contains a closed G-orbit.



χ-semi-invariants form a subspace $\mathbb{C}[X]_\chi$.

Form a ring $SI \coloneqq \bigoplus_\chi \mathbb{C}[X]_\chi$.

Lem. 9.4.1.

$SI = \mathbb{C}[V]^{[G,G]}$.

(G reductive, $[G,G]$ gen. by $ghg^{-1}h^{-1}$.)

Lem. 9.4.2. (Sato-Kimura)

If V has a dense open orbit, Then SI is poly. ring.

Quotient for quiver representations

$\theta \in \left(\mathbb{Z}^{V_0}\right)^*$. (Called weight.)

Rep. V is θ-s.s. if $\theta\left(\overrightarrow{\dim V}\right) = 0$ and $\theta\left(\overrightarrow{\dim V'}\right) \geq 0$ for $V' \subset V$.

θ-stable if further $\theta\left(\overrightarrow{\dim V'}\right) = 0 \Rightarrow V' = V$ or 0.

Want to identify with GIT stability.
Define

$$\chi_\theta : GL(\overrightarrow{\dim V}) \to \mathbb{C}^\times, \chi_\theta(g) = \prod_{v \in Q_0} \det(g_v)^{\theta_v}.$$

Note: $\Delta = \mathbb{C}^\times \in GL(\overrightarrow{\dim V})$ acts trivially.

$$\chi_\theta(c \cdot I) = c^{\left(\theta, \overrightarrow{\dim V}\right)} = 1.$$

One parameter $\lambda: \mathbb{C}^\times \to GL(\overrightarrow{\dim V})$ that has $\lim_{t\to 0} \lambda(t) \cdot V$ gives a filtration:

Weight decomposition

$$V_x = \bigoplus_{n\in\mathbb{Z}} V_x^{(n)}$$

where $\lambda(t)$ acts on $V_x^{(n)}$ as multi. by t^n.

Arrow: matrix $V_a = \left(V_a^{(m,n)}: V_{t_a}^{(n)} \to V_{h_a}^{(m)}\right)$.

$\lambda(t)$ acts on $V_a^{(m,n)}$ as multi. by t^{m-n}.

Has $\lim_{t\to 0}$ iff $V_a^{(m,n)} = 0$ for $m < n$, that is,

V_a preserves $V_x^{\geq n}$ $\forall n$, meaning

$V^{\geq n}$ forms subrepresentations.

$\cdots \supset V^{\geq n} \supset V^{\geq n+1} \supset \cdots$

$$\lim_{t\to 0} \lambda(t) \cdot V = \bigoplus_{n\in\mathbb{Z}} V^{\geq n}/V^{\geq n+1}.$$

Converse: a filtration always arise in this way (although such λ is not unique).

The filtration is trivial (meaning $V^{\geq n}$ are either 0 or V) implies

$\lambda(t)$ acts on V_x as $t^n \cdot Id$ (and same n for all x)

meaning $\lambda \subset \Delta$.

Prop. V is GIT χ_θ-semistable iff θ-semistable. (Similar for stable.)

Proof.

Recall the numerical criterion:

χ_θ-semistable iff

$\lim_{t\to 0} \lambda(t)\cdot x$ exists $\Rightarrow (\chi_\theta, \lambda) \geq 0$.

$\langle \chi_\theta, \lambda\rangle$ (t-power of $\chi_\theta(\lambda(t))\colon \mathbb{C}^\times \to \mathbb{C}^\times$) in terms of filtration $V^{\geq n}$:

$$\langle \chi_\theta, \lambda\rangle = \sum_x \theta(x) \sum_n n \dim V_x^{(n)} = \sum_n n\ \theta\left(\overrightarrow{\dim V^{\geq n}/V^{\geq n+1}}\right)$$

$$= \sum_n \theta\left(\overrightarrow{\dim V^{\geq n}}\right)$$

a finite sum since $\theta\left(\overrightarrow{\dim V}\right) = 0$ (and $V^{\geq n} = V\ \forall n \ll 0$).

θ-s.s.:

$\theta\left(\overrightarrow{\dim V'}\right) \geq 0\ \forall V' \subset V$.

θ-s.s. $\Rightarrow$ GIT χ_θ-s.s.:

$$\langle \chi_\theta, \lambda\rangle = \sum_n \theta\left(\overrightarrow{\dim V^{\geq n}}\right) \geq 0.$$

θ-stable $\Rightarrow$ GIT χ_θ-stable:

$\langle \chi_\theta, \lambda\rangle = 0$

$\Rightarrow \theta\left(\overrightarrow{\dim V^{\geq n}}\right) = 0\ \ \forall n$

$\Rightarrow V^{\geq n} = V$ or $0\ \forall n$

$\Rightarrow \lambda \in \Delta$.

GIT χ_θ-s.s. $\Rightarrow \theta$-s.s.:

For $V' \subset V$, take the filtration

$V \supset V' \supset 0$

and a corresponding one parameter λ.

GIT s.s. implies

$$\langle \chi_\theta, \lambda\rangle = \theta\left(\overrightarrow{dim\, V}\right) + \theta\left(\overrightarrow{dim\, V'}\right) = \theta\left(\overrightarrow{dim\, V'}\right) \geq 0.$$

GIT χ_θ-stable $\Rightarrow \theta$-stable:

If $\theta\left(\overrightarrow{dim\, V'}\right) = 0$,

$\langle \chi_\theta, \lambda \rangle = 0$ and hence $\lambda \subset \Delta$.
Thus the filtration is trivial and $V' = 0$ or V.

Recall: for V GIT s.s., $GL \cdot V$ is closed iff
$\lim_{t\to 0} \lambda(t) \cdot V$ exists and $(\chi, \lambda) = 0 \Rightarrow \lim_{t\to 0} \lambda(t) \cdot V \in G \cdot V$.
$(\chi, \lambda) = \sum_n \theta\left(\overrightarrow{\dim V^{\geq n}}\right) = 0$ iff
each $V^{\geq n}$ has $\theta = 0$, and hence s.s.
(Sub-rep. of $V^{\geq n}$ is sub-rep. of V and hence has $\theta \geq 0$.)
V is isom. to
$$\lim_{t\to 0} \lambda(t) \cdot V = \bigoplus_n V^{\geq n}/V^{\geq n+1}.$$
Jordan-Holder filtration (for Abel. cat. of s.s. rep.) exists,
where the graded pieces
$V^{\geq n}/V^{\geq n+1}$ are simple s.stable objects.
Simple s.s. $\Leftrightarrow$ stable:
$\rightarrow$)
If $V' \subset V$ has $\theta\left(\overrightarrow{dim\, V'}\right) = 0$, then V' is also s.s. and hence $= V$ or 0 since simple.
$\leftarrow$)
For $V' \subset V$ s.s., $\theta\left(\overrightarrow{dim\, V'}\right) = 0$ and by stable $V' = V$ or 0.

Thus we obtain:

Prop.
$GL \cdot V$ is closed in s.s. iff V is direct sum of stables.

(Direct sum of stables have direct sum of $c \cdot Id$ as stabilizers.)

Recall: GIT equiv. for s.s. objects $V \sim W$:
$\exists \lambda_1, \lambda_2$ *with* $\chi(\lambda_1) = \chi(\lambda_2) = 0$ *such that*
$\lim_{t\to 0} \lambda_1(t) \cdot V$ and $\lim_{t\to 0} \lambda_2(t) \cdot W$ belong to the same closed G-orbit.
By above prop., $\lim_{t\to 0} \lambda_1(t) \cdot V$ and $\lim_{t\to 0} \lambda_2(t) \cdot W$ are direct sums of the same stables.
Thus:

Prop.
GIT

s.s. $V \overset{GIT}{\sim} W$ iff V and W have the same graded pieces in Jordan-Holder filtration. (Called S-equiv.)

Finite dim. algebra

Corr. to quiver Q with relations:

Take a decomposition $A = P_1^{\oplus m_1} \oplus \cdots \oplus P_n^{\oplus m_n}$.

Take $P = P_1 \oplus \cdots \oplus P_n$.

$B = \mathrm{End}_A(P)^{op}$ is a basic alg. Morita equiv. to A (Ch. 3),

that is,

$B/\mathrm{rad}(B) \cong \mathbb{C}^n$ as algebra.

Define $\mathbb{C}^n$-bimod

$M := \mathrm{rad}(B)/\mathrm{rad}(B)^2$

which corr. to a quiver Q:

vertex set is the standard basis of $\mathbb{C}^n$

(which are indecomp. proj. mod. P_i of A);

arrow set is a basis of $e_i \cdot M \cdot e_j$.

Has surjective $\mathbb{C}Q \to B$ whose kernel is

admissible ideal J.

A-mod can be understood as subcat. of $\mathbb{C}Q$-mod

(that satisfies the relations):

the functor $Hom_A(-, M)$ restricted to cat. of proj. A-mod.

corr. to $\mathbb{C}Q$-mod. (Morita equiv.)

$M = Hom_A(A, M)$ is reconstructed from this functor.

Vertices of Q corr. to simple A-mod.

$K_0(A\text{-mod})$ is the free Abel. group gen. by Q_0

(by Jordan-Holder thm.).

Char. θ for A-mod is element of $\mathbb{Z}^{Q_0}$.

Using the above identification, get:

Thm. 4.1.

The GIT quotient $M_A(\alpha, \theta)$ gives the moduli space of θ-semistable A-mod. of dim. α. The points correspond to S-equiv. classes of θ-semistable A-mod.

Prop. 4.3.
$M_A(\alpha, \theta)$ is projective.

Proof.
$M_A(\alpha, \theta)$ is projective over $V_A(\alpha)//GL(\alpha)$ (character zero case).
For character zero,
all points are semi-stable.
For any point, the orbit closure must contain a closed orbit.
(If $G \cdot p$ not closed, has $p' \in \overline{G \cdot p} - G \cdot p$. Keep on doing this until getting a point with closed orbit.)
By prop. above, has closed orbit iff direct sum of stables, which are simple objects.
Thus the orbit closure must contain a semi-simple object, which is the unique direct sum of simple rep. over the vertices (in given α).
Thus all points in $V_A(\alpha)$ are equiv. (when $\theta = 0$) and hence
$V_A(\alpha)//GL(\alpha)$ is just a point.

Moduli space

It is pretty tautological that $M_A(\alpha, \theta)$ is a course moduli, namely
for a family of θ-s.stable A-mod over B, has a canonical map
$B \to M_A(\alpha, \theta)$
(choose trivialization of the vector bundles at vertices, and then have map to Rep_α).

Prop. 5.3.
If α is indivisible, then $M_A^s(\alpha, \theta)$ is a fine mod. of θ-stable A-mod.

Proof.
Want a taut. bundle over $M_A^s(\alpha, \theta)$ whose fiber over $[V]$ is V (equipped with $A \to End(V)$).
Take $\mathrm{Rep}_\alpha^s \times V_x$ for each vertex x, and take quotient by GL_α.
(Just usual quotient for stable points.)
TROUBLE:
$\Delta \subset GL_\alpha$ acts trivially on Rep_α^s, but acts on V_x by scaling!
Then $(\mathrm{Rep}_\alpha^s \times V_x)/GL_\alpha \to \mathrm{Rep}_\alpha^s/GL_\alpha$ is problematic!

Remedy:
Modify the GL_α-action on the second factor such that Δ acts trivially.
Take $(g \cdot p, \chi(g) \cdot g \cdot v) \in \mathrm{Rep}_\alpha^s \times V_x$
where $\chi: G \to \mathbb{C}^\times$ such that $\chi(c \cdot \mathrm{Id}_x) = c^{-1}$.
Character takes the form

$$\chi(-) = \prod_x \det(-)^{\psi_x}.$$

Then need $\sum \psi_x \alpha(x) = -1$
which exists iff α is indivisible.
Then done.

Rmk. 5.4.
For α indivisible, $M_A(\alpha, \theta) = M_A^s(\alpha, \theta)$ for generic θ.