Toric moduli of quiver rep.

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[Craw; Smith - Toric var. as fine moduli]

Toric var.

ex. $\mathbb{P}^2$.

Have fan (complex) and polytope (symplectic) description.

GIT quotient:

Each ray ρ is associated a variable x_ρ of $\mathbb{C}^m$.

From the primitive vectors of rays, get exact sequence

$0 \to K \to \mathbb{Z}^m \to N \to 0$.

Have complex subtorus $K_\mathbb{C}/K \subset (\mathbb{C}^\times)^m$ that acts on $\mathbb{C}^m$.

$\mathbb{C}^m//_\theta(\mathbb{C}^\times)^k$ where

unstable points are defined by the ideal gen. by monomials

$$x^{\hat{\sigma}} = \prod_{\rho\in\hat{\sigma}} x_\rho$$

where

$\hat{\sigma}$ is set of rays not contained in σ.

ex. $\mathbb{P}^2 = \mathbb{C}^3//_\theta\mathbb{C}^\times$.

The set of unstable points is

$\{x_1 = x_2 = x_3 = 0\} \subset \mathbb{C}^3$.

$\theta \in K^* = \mathrm{Pic}(Y)$ can also be understood as `level'

of $M_\mathbb{R} \subset (\mathbb{R}^m)^*$.

The dual sequence

$0 \to M \to (\mathbb{Z}^m)^* \to K^* \to 0$

has geometric meaning:

$0 \to M_Y \to \mathrm{CDiv}(Y) \to \mathrm{Pic}(Y) \to 0$.

Overview

- dim $= (1, \dots, 1)$ rep. of Q (without relations)
  defines a smooth proj. toric var.

- Embed a projective variety X to a quiver toric
  variety.
  (Generalization of Kodaira embedding to $\mathbb{P}^m$.)

Take line bundles
$L \coloneqq (O_X, L_1, \dots, L_r)$
over X.

Take endo. alg. of L
which corr. to a quiver Q (of sections).
Q is rooted.
$End(L) \cong \mathbb{C}Q/R$.

Denote the toric var. of dim=(1,...,1) rep. of Q by $|L|$.
Analogous to Kodaira embedding,
want to have $\phi: X \to |L|$.
Then X is identified as mod. of quiver rep. with
relations.

- This paper: do this for toric variety X.
  Have nice graph-theoretical interpretation.

Main Theorem 1.2:
Can find L such that ϕ is embedding,
and X is the (fine) moduli
$M_\theta(Q, R)$
of stable rep. (of dim=1) of (Q, R)
where R is the ideal of relations.

The universal rep. over $X \cong M_\theta(Q, R)$ decomposes into
taut. line bundles which equal to L_i.

Recall taut. bundles on quiver moduli

Have $M_\theta(\alpha)$ for $\theta \in \mathrm{Wt}(Q)$.
In this paper,
$\alpha = (1, \dots, 1)$. Primitive.
$G = \{(g_0, \dots, g_r) \in (\mathbb{C}^\times)^{Q_0} : g_0 = 1\}$.

(Kill the trivial action of overall scaling.
Can do this since Q is rooted.)

For generic θ, M_θ is fine moduli.
Have tautological line bundles F_i $\forall i \in Q_0$.
Just take trivial product $Rep_\alpha \times V_i$ and quotient by G.
F_0 is trivial since G acts trivially on 0th vertex.

Note:
This is to understand X as fine moduli of objects.
L may NOT be complete strong exceptional collection.
(No higher Ext, no morphism from later to earlier, self morphism space is 1d, gen. the category.)
(L provides an equiv. on der. cat.
iff it is such a collection.)

Every toric variety has
a complete exceptional collections of coherent sheaves, but
such collection of line bundles may not exist.
(Has counter-example in dim=2.)

(A-side analog: moduli of stable Lagrangians.)

<u>Toric var. associated to quiver</u>

Recall $G = \left\{(g_0, \dots, g_r) \in (\mathbb{C}^\times)^{Q_0} : g_0 = 1\right\}$ acts on
Rep_α by $g_{h_a} \cdot V(a) \cdot g_{t_a}^{-1}$.

Thus think of $G \subset (\mathbb{C}^\times)^{Q_1}$ induced by
$\mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_1}$
with ϵ_i mapped to f where
$f_a = 1, -1, 0$ if $h_a = i, t_a = i$, otherwise respectively.

Get the toric exact sequence.
Taking the dual, have:
$\mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_0}$
by

$\sum_a ((f_a)_{h_a} - (f_a)_{t_a})$.

The image lies in
weight lattice $\mathrm{Wt}(Q)$:
$\theta \in \mathbb{Z}^{Q_0}$ with $\Sigma\theta = 0$.
Rank $= |Q_0| - 1$.

Surjective to $\mathrm{Wt}(Q)$ for Q conn.
Define the kernel as $\mathrm{Cir}(Q)$.
Called circulation.

Thus have exact seq.
$0 \to \mathrm{Cir}(Q) \to \mathbb{Z}^{Q_1} \to \mathrm{Wt}(Q) \to 0$.
Rank of $\mathrm{Cir}(Q) = |Q_1| - |Q_0| + 1$.

Easy description of $\mathrm{Cir}(Q)$:
Given a walk γ (which is allowed to have inverse of arrows)
$f(\gamma) := \sum_a \mathrm{mult}_\gamma(a) \cdot a \in \mathbb{Z}^{Q_1}$.
$f(\gamma) \in \mathrm{Cir}(Q) \Leftrightarrow \gamma$ is closed.
$Cir(Q)$ are generated by such closed walks.
Rank$(N) = |Q_1| - |Q_0| + 1$.

The dual sequence is the toric exact sequence:
$0 \to (\mathrm{Wt}(Q))^* \to (\mathbb{Z}^{Q_1})^* \to N \to 0$.

Fan Σ_Q:
ray ρ_a for $a \in Q_1$ generated by $ev_a \in N_\mathbb{R} = \mathrm{Cir}(Q)^*$.
$\rho_{a_1}, \dots, \rho_{a_l}$ span a cone iff
exists spanning tree rooted at source of Q
which does not contain any of $a_1 \dots a_l$.
(Spanning tree has #edges = #vertices - 1.
Thus $l \leq |Q_1| - |Q_0| + 1$.)
Max. cone 1-1 corr. to
spanning tree rooted at source of Q.

Prop. 3.8.
Q conn. rooted acyclic.

The fine mod. M_θ for any rational θ in the GIT chamber containing

$$\theta_0 \coloneqq \sum_{i \in Q_0} (\chi_i - \chi_0)$$

($\chi_i \in \mathbb{Z}^{Q_0}$ is std basis)
($\theta_0 \perp \alpha = (1, \dots, 1)$)
is the toric var. defined by Σ_Q,
which is unimod. and proj.

It is the GIT quot.
$\mathrm{Spec}(\mathbb{C}[y_a : a \in Q_1]) - V(B_Y)$
by $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Wt}(Q), \mathbb{C}^\times)$,
B_Y gen. by all the monomials

$$\prod_{a \in \text{rooted spanning tree}} y_a$$

which can also be written as intersection of ideals
$\cap_{i=1}^{r} (y_a : h_a = i)$.
(Set of vertices labeled by $0, \dots, r$.)
(For each vertex other than source, choose an arrow going to that vertex. This forms a spanning tree.)

Conclusion.
The sequence
$0 \to \mathrm{Cir}(Q) \to \mathbb{Z}^{Q_1} \to \mathrm{Wt}(Q) \to 0$
is isom. to
$0 \to M_Y \to \mathrm{CDiv}(Y) \to \mathrm{Pic}(Y) \to 0$.
(Cartier div. up to principal div.)

The dual sequence
$0 \to (\mathrm{Wt}(Q))^* \to (\mathbb{Z}^{Q_1})^* \to (\mathrm{Cir}(Q))^* \to 0$
tells us the rays of the fan in $N = (\mathrm{Cir}(Q))^*$.
(Counting the multiplicity of an arrow appearing in a circulation.)

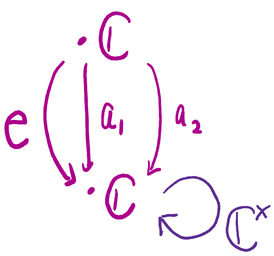
The irrelevant ideal is gen. by monomials

$$\prod_{a \in \text{rooted spanning tree}} y_a \,.$$

(Rooted spanning tree is formed by choice of an arrow heading at each vertex other than root.)

Thus each rooted spanning tree corresponds to a max. cone of the fan,
gen. by the rays which are arrows NOT contained in the tree.

Ex. 3.10.

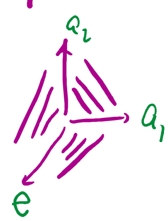


$$\mathbb{Z}^{Q_1} \to \mathrm{Cir}^* = \langle a_1 e^{-1}, a_2 e^{-1} \rangle^*$$

$a_1 \longmapsto (1,0)$

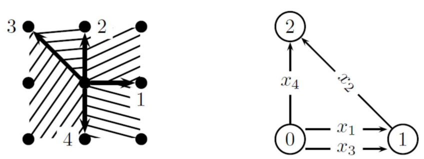
$a_2 \longmapsto (0,1)$

$e \longmapsto (-1,-1)$



$B_Y = (y_{a_0}, \dots, y_{a_m})$
(monomials corr. to spanning trees).
$Y = \mathbb{P}^m$.
If Q is the quiver of sections of $\{O_X, L\}$, then this is the usual linear system $|L|$ of a variety X.

Ex. 3.11.



(a) Fan (b) Quiver of sections

Figure 1. Hirzebruch surface $\mathbb{F}_1$.

Basis of $\mathrm{Cir}(Q)$ (corr. to mero. functions):
$(a_1 a_3^{-1}, a_3 a_2 a_4^{-1})$
Rays and toric divisors correspond to a_i.
a_1: (1,0); a_2: (0,1); a_3: (−1,1), a_4: (0,−1).

$S_Y = \mathbb{C}[y_1, y_2, y_3, y_4]$;
$B_Y = (y_1, y_3) \cap (y_2, y_4)$.
$G = (\mathbb{C}^\times)^2$.
$Y = \mathbb{F}_1$.

Nef cone:
(divisors that has non-negative intersections with all effective curves)

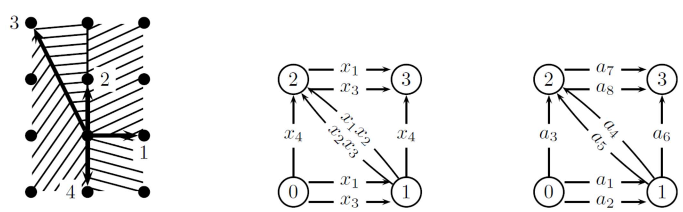
The above GIT chamber in $\mathrm{Wt}(Q) \otimes \mathbb{R}$ is the ample cone of $\mathbb{Q}$-div. classes.
($O_Y(\theta_0)$ is very ample.)
The nef cone is the closure

$$\mathrm{Nef}_{\mathbb{Q}}(Y) = \bigcap_{Q' \subseteq Q} \left\{ \sum_{a \in Q'_1} \lambda_a [D_a] : \lambda_a \in \mathbb{Q}_{\geq 0} \right\},$$

where Q' is rooted spanning tree,
D_a are toric divisors.
(Spanning trees are corr. to max. toric charts.
The above are divisors that can be `moved away' from any given max. chart,
and hence must intersect non-neg. with any toric curve,
which must be `Zariski contained' in a max. chart.)

In the example of $\mathbb{P}^n$, this is non-neg. comb. of the toric div. D_i.

Ex. 3.12.



(a) Fan (b) Quiver of sections (c) Listing the arrows

Figure 2. Hirzebruch surface $\mathbb{F}_2$.

Basis of $\mathrm{Cir}(Q)$ (corr. to mero. functions):
$(a_1a_2^{-1}, a_1a_4a_3^{-1}, a_1a_5a_3^{-1}, a_1a_6a_7^{-1}a_3^{-1}, a_1a_6a_8^{-1}a_3^{-1})$

$D_1 \sim D_2, D_3 \sim D_1 + D_4 \sim D_1 + D_5,$
$D_1 + D_6 \sim D_3 + D_7 \sim D_3 + D_8.$
Then $D_4 \sim D_5, D_7 \sim D_8, D_6 \sim D_4 + D_7$.
Thus $\mathrm{Pic}(Y)$ gen. by (image of)
$D_1, D_4, D_7 \in \mathrm{CDiv}(Y) \to \mathrm{Pic}(Y)$.

$S_Y = \mathbb{C}[y_1, \ldots, y_8];$
$B_Y = (y_1, y_2) \cap (y_3, y_4, y_5) \cap (y_6, y_7, y_8).$
$G = (\mathbb{C}^\times)^3.$

Y is 5 dim. toric with 8 toric divisors.
Toric fixed points (max. cones) corr. to
$2 \times 3 \times 3 = 18$
spanning trees.

$$\mathrm{Nef}_{\mathbb{Q}}(Y) = \bigcap_{Q' \subseteq Q} \left\{ \sum_{a \in Q'_1} \lambda_a [D_a] : \ \lambda_a \in \mathbb{Q}_{\geq 0} \right\},$$

Express $h_1 D_1 + h_2 D_4 + h_3 D_7$ in divisors involved in spanning trees to get conditions on h_i:
(moving it away from given max. toric chart)
$h_1 D_1 + h_2 D_4 + h_3 D_7$ (147)
$= h_1 D_1 + h_2(D_3 - D_1) + h_3 D_7$ (137)
$= h_1 D_1 + h_2 D_4 + h_3(D_6 - D_4)$ (146)
$= h_1 D_1 + h_2(D_3 - D_1) + h_3(D_6 - D_3 + D_1)$ (136)
Thus
$h_1 \geq h_2 \geq h_3 \geq 0.$

$h_1 D_1 + h_2 D_4 + h_3 D_7$ expressed as weights:
($\mathrm{CDiv}(Y) \to \mathrm{Pic}(Y)$ identified with $\mathbb{Z}^{Q_1} \to \mathrm{Wt}(Q)$)
(#head - #tails at each vertex)
$p_0 = -h_1, p_1 = h_1 - h_2, p_2 = h_2 - h_3, p_3 = h_3.$

Thus nef cone is
$\{p_0 + p_1 + p_2 + p_3 = 0, p_1 \geq 0, p_2 \geq 0, p_3 \geq 0\}.$

ex. anti-can. div.

$$\sum_{i=1}^{8} D_i \mapsto (-3, -1, 1, 3) \notin \text{nef cone}.$$

<u>Quiver of sections of toric</u>
Take distinct toric line bundle L_i. $L_0 = O_X$.
Indecomp. (T-inv.) section $s \in H^0(X, L_j \otimes L_i^{-1})$:
the divisor $div(s)$ cannot be sum $div(s') + div(s'')$
for $s' \in H^0(X, L_k \otimes L_i^{-1}), s'' \in H^0(X, L_j \otimes L_k^{-1})$.

Quiver of sections:
vertices 1-1 corr. to L_i;
arrows 1-1 corr. to T-inv. indecomp. sections (up to

scaling).
(Recall: T-inv. hol. sections corr. to certain lattice points in M.)

Hence paths give non-zero elements in $H^0(L_i, L_j)$.
If projective, one of $H^0(L_i, L_j), H^0(L_j, L_i)$ is zero (for $i \neq j$).
Hence must be acyclic.
Reorder such that $H^0(X, L_j \otimes L_i^{-1}) = 0$ if $j < i$.

Assume taken $H^0(X, L_i) \neq 0\ \forall i$.
Then Q connected and rooted at vertex 0.

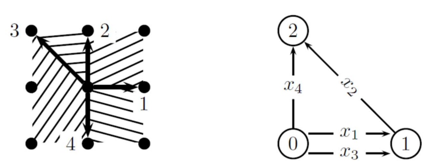
Ideal of relations gen. by
$p - p' \in \mathbb{C}Q$
where $\mathrm{div}(p) = \mathrm{div}(p')$, same head and tail.

Note: two non-zero sections define the same div. iff they are proportional (by compactness).

By def.,
Prop. 3.3.
$\mathbb{C}Q/R$ is isom. to $\mathrm{End}(\bigoplus L_i)$.

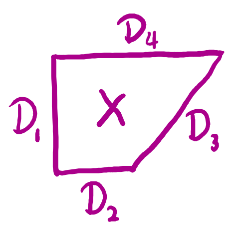
Geometry ⟷ Quiver Rep. theory

Ex. 3.6.



(a) Fan (b) Quiver of sections

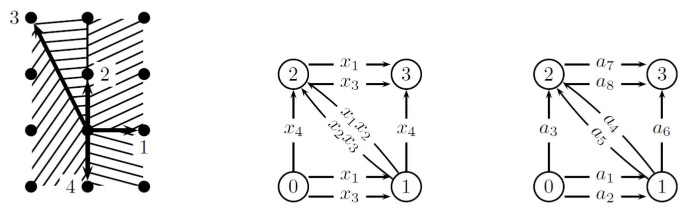
Figure 1. Hirzebruch surface $\mathbb{F}_1$.



D_1 : $(f) \geq -D_1$: $\{1, z_1^{-1}\} + D_1$: D_1, D_3.
$D_1 \to D_4$: " $\geq -D_4 + D_1$: $\{z_1 z_2\} + D_4 - D_1$: D_2.
D_4 : " $\geq -D_4$: $\{1\} + D_4$: D_4.
($z_2, z_1 z_2$ are NOT indecomp.)

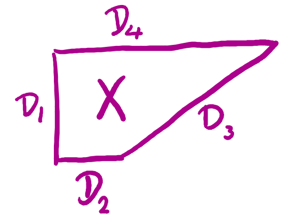
Take $L = (\mathcal{O}_X, \mathcal{O}_X(D_1), \mathcal{O}_X(D_4))$.
Ideal of relations is trivial.

Ex. 3.7.



(a) Fan (b) Quiver of sections (c) Listing the arrows

Figure 2. Hirzebruch surface $\mathbb{F}_2$.



Take $L = (O_X, O_X(D_1), O_X(D_4), O_X(D_1 + D_4))$.
$R = (a_2a_4 - a_1a_5, a_4a_8 - a_5a_7, a_2a_6 - a_3a_8, a_1a_6 - a_3a_7)$.

D_1 : $(f) \geq -D_1$: $1, z_1^{-1}$ D_1, D_3

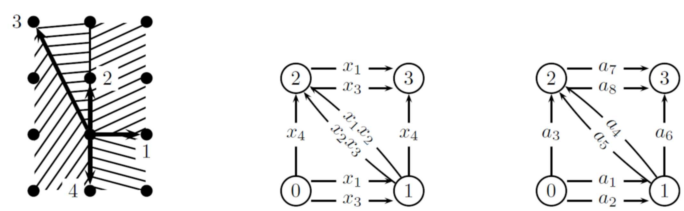
$D_1 \to D_4$: .. $\geq -D_4 + D_1$: $z_1z_2, z_1^2z_2$ $D_3 + D_2$, $D_1 + D_2$

D_4 : .. $\geq -D_4$: 1. D_4

⋮

Conclude:
From toric X,
choose $L = \{L_i : i = 0 \ldots r\}$ to get
(complete) quiver of sections Q.
From Q, get toric
Y which is also denoted as
$|L|$.

Morphism $X \to Y$



(a) Fan (b) Quiver of sections (c) Listing the arrows

Figure 2. Hirzebruch surface $\mathbb{F}_2$.

On total coord. ring:
$\Phi_Q : S_Y \to S_X$:
$y_a \mapsto x^{\mathrm{div}(a)}$.
(Take toric divisor defined by toric holo. section.
The multiplicity is positive.)

Prop. 4.1 (Criterion for base-point free):
Have $X \to Y$ iff
$B_X \subset \mathrm{rad}(B_Q)$
$(B_Q \coloneqq S_X \cdot \Phi_Q(B_Y))$
iff
For each cone σ of fan of X, has spanning tree Q'
rooted at 0 such that

$\mathrm{supp}(\mathrm{div}(a)) \subset \hat{\sigma}\ \forall a \in Q_1'$.
($\hat{\sigma}$ is the set of rays not contained in σ.
The removed locus in def. of X corr. to ideal gen. by the monomials $x^{\hat{\sigma}}\ \forall \sigma$.)

Proof.
The first is by def. and group equivariance.
For second equiv.,
recall B_Y is gen. by monomials $y_{a_k} \dots y_{a_1}$ of rooted spanning trees.
$\Phi_Q(y_{a_k} \cdots y_{a_1}) = x^{\mathrm{div}(a_k)} \dots x^{\mathrm{div}(a_1)}$.
By multiplying more $x \in S_X$ and then taking roots,
$\mathrm{rad}(B_Q)$ is gen. (over S_X) by $\prod_{i \in \cup_{l=1}^k \mathrm{supp}(\mathrm{div}(a_l))} x_i$.

B_X gen. by $x^{\hat{\sigma}}$ for all σ.
$B_X \subset \mathrm{rad}(B_Q)$ iff
for all σ, has certain rooted spanning tree such that $x^{\hat{\sigma}}$ has factor $\prod_{i \in \cup_{l=1}^k \mathrm{supp}(\mathrm{div}(a_l))} x_i$.

Corr. 4.2.
A (complete) quiver of sections is basepoint free iff all line bundles involved are basepoint free.

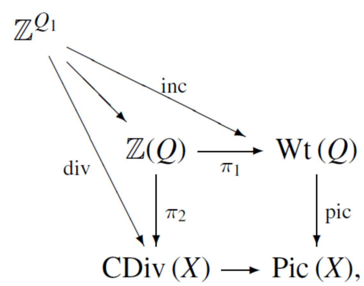
Proof.
→)
For each cone σ of fan of X, has spanning tree Q' rooted at 0 such that
$\mathrm{supp}(\mathrm{div}(a)) \subset \hat{\sigma}\ \forall a \in Q_1'$.
Then for each vertex, has path (in Q') from 0 to that vertex such that $\mathrm{supp}(\mathrm{div}(a)) \subset \hat{\sigma}$ for all arrows in that path.
Then the corresponding section is non-zero at the corresponding toric strata.
←)
Given a cone of fan,
has path from 0 to every vertex satisfying the above condition $\mathrm{supp}(\mathrm{div}(a)) \subset \hat{\sigma}$.
Suppose already fix a tree containing vertex $0, \dots, k$ satisfying the condition.
For the vertex $k+1$,

has a path satisfying the condition.
If the path does not pass through any of $0, \ldots, k$,
simply add this path to the tree.
Otherwise,
consider the last time that the path pass thru $0, \ldots, k$.
Only take the part of the path after that vertex and add
this to the tree.

Section lattice $\mathbb{Z}(Q)$:
Image of

$$\left(\sum_a ((f_a)_{h_a} - (f_a)_{t_a}), \sum_a f_a \, div(a)\right):$$

$\pi: \mathbb{Z}^{Q_1} \to \mathrm{Wt}(Q) \oplus CDiv(X)$.



$\mathbb{N}^{Q_1}$ corr. to monomials.
For two monomials y^u and y^v,
$u - v \in \mathrm{Ker}(\pi) \cap \mathbb{N}^{Q_1}$ means:
u and v have the same weights (homogeneous);
y^u and y^v equal after pullback to X.

Thus the image is defined by the ideal
$I_Q \coloneqq \left(y^u - y^v : u - v \in \mathrm{Ker}(\pi) \cap \mathbb{N}^{Q_1}\right)$.

Prop. 4.3.
The image of $X \to Y$ given by a basepoint-free quiver
of sections is $V(I_Q)//_\theta G$, that is,
$\left(V(I_Q) - V(B_Y)\right)//G$.

The following gives useful way to find L_i to have an
embedding.

Cor. 4.10.
Let $L = \bigotimes L_i$. Assume

$H^0(L_1) \otimes \cdots \otimes H^0(L_r) \to H^0(L)$
is surjective. Then
$X \to Y$ is a closed embedding iff L is very ample.

Prop. 4.14.
Let $L_1, \dots, L_{r-1}$ be basepoint free.
If $L_1, \dots, L_{r-1}$ generates an ample line bundle,
then there exists L_r such that
$(O_X, L_1, \dots, L_r)$
gives an embedding.