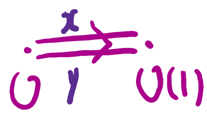
Hochschild cohomology for quiver algebra

Monday, January 20, 2020 2:59 PM

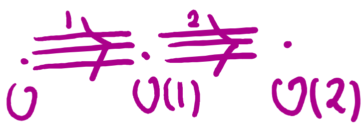
[Bergman - Deformations and D-branes]

For Fano and local CY, may find A such that
$D(X) \cong D(A)$.
[Bondal-Orlov]: complete strong exceptional collection of sheaves E_i.
Then $Hom_{D(X)}(E,-)$ and $E \otimes_A -$
gives the equivalence.

ex. $\mathbb{P}^1$.



ex. $\mathbb{P}^2$.



$x_1 y_2 \sim y_1 x_2$
$\vdots$

ex. $O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1)$.
[Van den Bergh]

$xyz \sim zyx$
$\vdots$

ex. $K_{\mathbb{P}^2}$.

$x_1 y_2 \sim y_1 x_2$
$\vdots$

Hochschild-Kostant-Rosenberg Theorem:

$$\bigoplus_{j+k=i} H^j(X, \wedge^k TX) \cong HH^i(D(X)).$$

Deformation theory:
$HH^i(B) = \mathrm{Nat}(Id_B, [i])$.

For algebra A (and B),

functor $F: D(A) \to D(B)$ given by
$A - B$ bimod. M.
$F(-) = - \otimes_A^L M$.
In particular Id_A is realized by A as $A - A$ bimod.

$HH^i(D(A)) = \mathrm{Hom}_{D(A-A)}(A, A[i]) = \mathrm{Ext}^i_{A-A}(A, A)$.

Bar resolution for the A-bimod. A:
$\ldots \to A^{\otimes_k 4} \to A^{\otimes_k 3} \to A^{\otimes_k 2} \to 0$.
Differential:

$$a_1 \otimes \cdots \otimes a_n \mapsto \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n\,.$$

Then take
$\mathrm{Hom}_{A-A}(A^{\otimes p}, A) = \mathrm{Hom}_k(A^{\otimes(p-2)}, A)$.
Get
$0 \to A \to \mathrm{Hom}_k(A, A) \to \mathrm{Hom}_k(A \otimes_k A, A) \to \cdots$
Differential:

$$\begin{aligned}(\delta f)(a_1, \ldots, a_{n+1}) = & a_1 f(a_2, \ldots, a_{n+1}) + \sum_i (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ & + (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1}\,.\end{aligned}$$

Then take cohomology.

$(\delta f_0)(a) = a \cdot f_0 - f_0 \cdot a$

$(\delta f_1)(a, b) = a \cdot f_1(b) - f_1(ab) + f_1(a) \cdot b$

$(\delta f_2)(a, b, c) = a f_2(b, c) - f_2(ab, c) + f_2(a, bc) - f_2(a, b) \cdot c$

$HH^0(A)$ equals to center of A.
$HH^1(A)$: outer derivations.
$HH^2(A)$: 1st order term of formal deformations

$$\mu(a,b) = m(a,b) + \sum_{i=1}^{\infty} \hbar^i \mu_i(a,b) \ .$$

Associativity implies $\delta\mu_1 = 0$.
Obstruction (to extend to higher $\hbar$-order) in $HH^3(A)$.

<u>Projective resolution for quiver algebra</u>

Now take
$A \coloneqq \mathbb{C}Q/I$.
Choose basis of admissible minimal relations: $R \in I/(IJ + JI)$.
(J is the ideal of non-trivial paths.)
Have unique source and target for each relation.

To resolve A as A-bimod, take proj.
$P_{ij} \coloneqq Ae_i \otimes_{\mathbb{C}} e_j A$.

$$\dots \to \bigoplus_R P_{t(R)s(R)} \xrightarrow{g} \bigoplus_a P_{t(a)s(a)} \xrightarrow{f} \bigoplus_i P_{ii} \xrightarrow{m} A \to 0.$$

$$f\big((p \otimes p')_a\big) = pa \otimes p' - p \otimes ap' = \sum_x p \cdot `\partial_{e_x} a' \cdot p' \,.$$

$$g\big((p \otimes p')_R\big) = \sum_a p \cdot \partial_a R \cdot p'$$

where $\partial_a R \in P_{t(a)s(a)}$ (NOTE the sense of differentiation here).

The next term concerns relations among relations.
The differential is defined in a similar way.

ex.

Relation:

$c \sim ba \quad (R_1)$

$b \sim d \quad (R_2)$

$c \sim da \quad (R_3)$

Relation among relations:

$RR = R_1 + R_2 a - R_3 \sim 0$

(Note that still has unique head and tail)

$$P_{t(RR)s(RR)} \to \bigoplus_{R}^{3} P_{t(R)s(R)} \xrightarrow{g} \bigoplus_{a}^{4} P_{t(a)s(a)} \xrightarrow{f} \bigoplus_{i=1}^{3} P_{ii} \xrightarrow{m} A \to 0.$$

ex. $(v \otimes u)_a \overset{f}{\longmapsto} a \otimes u - v \otimes a \overset{m}{\longmapsto} 0$

$\partial_a R_1 = -b \otimes 1$

$\vdots$

$(w \otimes u)_1 \overset{g}{\longmapsto} (\overset{b}{-wb \otimes u}, -w \otimes au, \overset{c}{w \otimes u}, \overset{d}{0}) \longmapsto \boxed{-wba \otimes u} + wb \otimes au$

$-wb \otimes au + w \otimes bau$

$\boxed{+wc \otimes u} - w \otimes cu$

$(w \otimes v)_2 \longmapsto (0, w \otimes v, 0, -w \otimes v)$

$(w \otimes u)_3 \longmapsto (-wd \otimes u, 0, w \otimes u, -w \otimes au).$

$$P_{t(RR)s(RR)} \to \bigoplus_{R} P_{t(R)s(R)}$$

$$(w \otimes u)_{RR} \mapsto \sum_{R} w \cdot \partial_R RR \cdot u$$

$(-wb \otimes u, -w \otimes au, w \otimes u, 0)$

$$= (w \otimes u)_{R_1} + (w \otimes au)_{R_2} - (w \otimes u)_{R_3}. \longmapsto + (0, w \otimes au, 0, -w \otimes au)$$

$-(-wd \otimes u, 0, w \otimes u, -w \otimes au)$

$= 0.$

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$$\ldots \to \bigoplus_{R}^{3} P_{t(R)s(R)} \xrightarrow{g} \bigoplus_{a}^{4} P_{t(a)s(a)} \xrightarrow{f} \bigoplus_{i=1}^{3} P_{ii} \xrightarrow{m} A \to 0.$$

Projective resolution of left mod. V: right tensoring V,

$$\ldots \to \bigoplus_{R} P_{t_R} \otimes_{\mathbb{C}} V(s_R) \xrightarrow{g} \bigoplus_{a} P_{t_a} \otimes_{\mathbb{C}} V(s_a) \xrightarrow{f} \bigoplus_{i} P_i \otimes_{\mathbb{C}} V(i) \to V \to 0$$

where

$f(p \otimes v)_a = (p \cdot a) \otimes v - p \otimes (a \cdot v)$;

$g\big((p \otimes v)_R\big) = \sum_a p \cdot \partial_a R \cdot v$.

(Generalize quiver without relations we talked about before.)
Use it to compute $H^i(V, W)$.

ex. Take $V = S_i$:

$$\ldots \to \bigoplus_{R: s(R)=i} P_{t(R)} \xrightarrow{g} \bigoplus_{a: s(a)=i} P_{t(a)} \xrightarrow{f} P_i \to S_i \to 0$$

where

$f\big((p)_a\big) = pa$;

$g\big((p)_R\big) = \sum_{s(a)=i} p \cdot \partial_a^{\text{last}} R$.

ASSUME global dim. of A is 2.
(No relations among the chosen relations.)
Then no more term on the left.

Apply $\mathrm{Hom}(-, S_j)$. $\left(\mathrm{Hom}\big(P_i, S_j\big) = 0 \text{ if } i \neq j.\right)$

$i \neq j$:

$$0 \to 0 \to \bigoplus_{a:i\to j} \mathrm{Hom}\big(P_j, S_j\big) \xrightarrow{g^*} \bigoplus_{R:i\to j} \mathrm{Hom}\big(P_j, S_j\big) \to 0$$

$\mathrm{Hom}\big(P_j, S_j\big) = \mathbb{C}$. (Any $\gamma \cdot e_j$ is mapped to 0 unless $\gamma = 1$.)

$0 \to 0 \to \mathbb{C}^{a:i\to j} \to \mathbb{C}^{R:i\to j} \to 0$
$g^* = 0$:
For $\phi \in \bigoplus_{a:i\to j} \mathrm{Hom}(P_j, S_j)$,
$$\phi\left(g\left((p)_R\right)\right) = \phi\left(\sum_{a:i\to j} p \cdot \partial_a^{\text{last}} R\right) = 0$$
if we assume R has at least length two.
Hence
$H^0(S_i, S_j) = 0, H^1(S_i, S_j) = \mathbb{C}^{a:i\to j}, H^2(S_i, S_j) = \mathbb{C}^{R:i\to j}$

$i = j$:
$$0 \to \mathrm{Hom}(P_j, S_j) \xrightarrow{f^*} \bigoplus_{a:j\to j} \mathrm{Hom}(P_j, S_j) \xrightarrow{g^*} \bigoplus_{R:j\to j} \mathrm{Hom}(P_j, S_j) \to 0.$$
$f^* = 0$:
For $\phi \in \mathrm{Hom}(P_j, S_j)$,
$$\phi\left(f\left((p)_a\right)\right) = \phi(pa) = 0.$$
(ϕ maps anything other than e_j to 0.)
Hence
$H^0(S_j, S_j) = \mathbb{C}, H^1(S_j, S_j) = \mathbb{C}^{a:j\to j}, H^2(S_j, S_j) = \mathbb{C}^{R:j\to j}$.

Write $S = \bigoplus S_i$. Then we can summarize it as:

$$\bigoplus_{i=1}^{\infty} \mathrm{Ext}^1(S,S)^{\otimes i} \to J^\vee \ ,$$
$$\mathrm{Ext}^2(S,S) \xrightarrow{\sim} (I/(IJ+JI))^\vee$$

where J is the ideal of non-trivial paths;
$I/(IJ + JI)$ is the vector space spanned by minimal relations R.

<u>**CY3**</u>

Suppose $A = \mathbb{C}Q/I$, and we have a basis of minimal relations such that there is no relation among relations.

(Q has no oriented cycle.)

Add an arrow r to Q for each relation R, with the same source and target.
Get a new quiver $\bar{Q}$.

$$W = \sum_R r \cdot R$$

which are simple loops in $\bar{Q}$.

Relations for $\bar{Q}$:
$R_a := \partial_a W \ \ \forall a \in \overline{Q_1}$.
This is cyclic differentiation.

$$B := \mathbb{C} \cdot \bar{Q} / \langle \partial_a W : a \in \overline{Q_1} \rangle.$$

Relations among relations:
For each vertex $i \in \overline{Q_0}$,

$$RR_i := \sum_a R_a \cdot a \cdot e_i - \sum_a e_i \cdot a \cdot R_a \, .$$

These are loops in W
(up to cyclic permutation)
that are based at i.

$$\partial_{R_a} RR_i = e_{t_a} \otimes a e_i - e_i a \otimes e_{s_a}.$$

Hence resolution of B is given by:

$$0 \to \bigoplus_i P_{ii} \xrightarrow{h} \bigoplus_a P_{s(a)t(a)} \xrightarrow{g} \bigoplus_a P_{t(a)s(a)} \xrightarrow{f} \bigoplus_i P_{ii} \xrightarrow{m} B \to 0$$

where

$$h(pe_i \otimes e_i p') = \sum_a (p \otimes a e_i p' - pe_i a \otimes p')_a$$

$$= \sum_{s(a)=i} (p \otimes ap')_a - \sum_{t(a)=i} (pa \otimes p')_a .$$

Recall

$$f\big((p \otimes p')_a\big) = (pa \otimes p')_{s(a)} - (p \otimes ap')_{t(a)}.$$

$$g\big((p \otimes p')_a\big) = \sum_{a'} p \cdot \partial_{a'} \partial_a W \cdot p' .$$

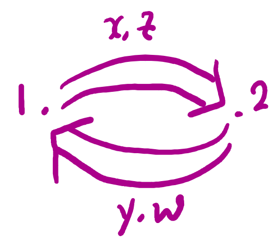
(Each arrow corresponds to one relation $\partial_a W$.)

ex.

$W = xyzw - zyxw.$

$\partial_x W = yzw - wzy.$

$\partial_y \partial_x W = (e_1 \otimes zw)_y - (wz \otimes e_2)_y.$



To compute HH, take $Hom_{B-B}(-, B)$.

$$0 \to \bigoplus_i \mathrm{Hom}_B(P_i, P_i) \xrightarrow{f^*} \bigoplus_a \mathrm{Hom}_B\big(P_{t(a)}, P_{s(a)}\big)$$

$$\xrightarrow{g^*} \bigoplus_a \mathrm{Hom}_B\big(P_{s(a)}, P_{t(a)}\big) \xrightarrow{h^*} \bigoplus_i \mathrm{Hom}_B(P_i, P_i) \to 0.$$

$f^*(\phi)$:

$$\big(f^*(\phi)\big)_a p = \phi\left(f\left(\big(p \otimes e_{s(a)}\big)_a\right)\right) = \phi\big(pa \otimes e_{s(a)} - p \otimes a\big)$$

$$= \phi(pa) - \phi(p) \cdot a.$$

Regard ϕ as sum of paths $i \to i$ (and hom is right multi.),

$$f^* \phi = \sum_a a\phi - \phi a .$$

$g^*(\phi)$:

$$\left(g^*(\phi)\right)_a p = \phi\left(g\left(\left(p \otimes e_{t(a)}\right)_a\right)\right) = \sum_{a'} \phi\left(\left(p \cdot \partial_{a'} \partial_a W\right)_{a'}\right).$$

Regard $\phi = \sum \phi_{a'}$ as paths $s(a') \to t(a')$,

$$g^*(\phi) = \sum_{a,a'} W^{(1)}_{aa'} \cdot \phi_{a'} \cdot W^{(2)}_{aa'}$$

where $\partial_a W = W^{(1)}_{aa'} \otimes W^{(2)}_{aa'}$ (indeed lin. comb.).

$h^*(\phi)$:

$$\left(h^*(\phi)\right)_i p = \phi\left(h\left((p \otimes e_i)_i\right)\right)$$

$$= \sum_{s(a)=i} \phi\left((p \otimes a)_a\right) - \sum_{t(a)=i} \phi\left((pa \otimes e_i)_a\right)$$

$$= \sum_{s(a)=i} \phi_a(p) \cdot a - \sum_{t(a)=i} \phi_a(pa)\,.$$

Regard $\phi = \sum \phi_a$ as paths $t(a) \to s(a)$,

$$h^*(\phi) = \sum_a \phi \cdot a - a \cdot \phi\,.$$

$HH^2(B)$:

To get element in $\bigoplus_a \mathrm{Hom}_B\left(P_{s(a)}, P_{t(a)}\right)$, take loop l and take

$$\sum_a (\partial_a l)_a$$

(paths $t(a) \to s(a)$).

Closed: $\sum_a (\partial_a l \cdot a - a \cdot \partial_a l) = 0$ since loop.

(Copy l cyclically; pos. when a appears on right; neg. when a appears on left.)

Called **superpotential deformations.**

Exact element:

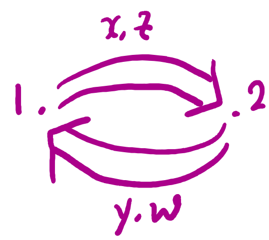
$$\sum_{a,a'} W^{(1)}_{aa'} \cdot \phi_{a'} \cdot W^{(2)}_{aa'}$$

for paths $\phi_{a'} : s(a') \to t(a')$.
In particular, if we take $\phi_{a'} = a'$, get

$$\sum_{a,a'} W^{(1)}_{aa'} \cdot a' \cdot W^{(2)}_{aa'} = \sum_{a} \partial_a W \, .$$

(Corr. to adding $\hbar W$ to W which is equiv. to original one.)

ex. Same W as above. Take
$l = xyzw$.



$$\sum_a \partial_a l \in \bigoplus_a \mathrm{Hom}_B\big(P_{s(a)}, P_{t(a)}\big):$$

$(\underset{x}{yzw}, \underset{y}{zwx}, \underset{z}{wxy}, \underset{w}{xyz})$. Adding these terms to the relations

Gives
$W_\hbar = (1+\hbar)xyzw - zyxw.$

$$0 \to \bigoplus_i \mathrm{Hom}_B(P_i, P_i) \xrightarrow{f^*} \bigoplus_a \mathrm{Hom}_B\big(P_{t(a)}, P_{s(a)}\big)$$

$$\xrightarrow{g^*} \bigoplus_a \mathrm{Hom}_B\big(P_{s(a)}, P_{t(a)}\big) \xrightarrow{h^*} \bigoplus_i \mathrm{Hom}_B(P_i, P_i) \to 0.$$

$$0 \to A_{①} \oplus A_{②} \xrightarrow{f^*} \left(A_{①} \cdot \langle y, w\rangle\right)^{\oplus 2}_{y,w} \oplus \left(A_{②} \cdot \langle x, z\rangle\right)^{\oplus 2}_{x,z}$$

$$\xrightarrow{g^*} \left(A_{2}\cdot\langle x,z\rangle\right)^{\oplus 2}_{y,w} \oplus \left(A_{1}\cdot\langle y,w\rangle\right)^{\oplus 2}_{x,z} \xrightarrow{h^*} \overbrace{A_{1}}^{\text{long loops at }1} \oplus A_{2} \longrightarrow 0$$

$$a \left| \begin{array}{c|c} & \xrightarrow{a'} \\ \begin{array}{c} y \\ w \\ x \\ z \end{array} & \begin{pmatrix} 0 & z(\ )x - x(\ )z & zw(\ ) - (\ )wz & (\ )wx - xw(\ ) \\ x(\ )z - z(\ )x & 0 & (\ )yz - zy(\ ) & zy(\ ) - (\ )zy \\ (\ )zw - wz(\ ) & yz(\ ) - (\ )zy & 0 & y(\ )w - w(\ )y \\ wx(\ ) - (\ )xw & (\ )xy - yx(\ ) & w(\ )y - y(\ )w & 0 \end{pmatrix} \end{array} \right| \qquad \left(\ \ \right) \begin{array}{c} y \\ w \\ x \\ z \end{array}$$

$$\left(\left(\overline{y\ \ w\ \ x\ \ z} \right) \begin{pmatrix} 0 & y \\ 0 & w \\ x & 0 \\ z & 0 \end{pmatrix}^t - \begin{pmatrix} y & w & 0 & 0 \\ 0 & 0 & x & z \end{pmatrix} \right) \left| \ \ \right| \begin{array}{c} y \\ w \\ x \\ z \end{array}$$

$$f^*\phi = \sum_a a\phi - \phi a\,.$$

$$g^*(\phi) = \sum_{a,a'} W^{(1)}_{aa'} \cdot \phi_{a'} \cdot W^{(2)}_{aa'}$$

$$h^*(\phi) = \sum_a \phi \cdot a - a \cdot \phi\,.$$

g^* means for each relation, replace one arrow by a possibly longer path, and sum up like Leibnitz rule.

In general, the second last term is direct sum over relations. Thus HH^2 is recording deformation of relations