Topology of Framed quiver moduli

Friday, February 28, 2020 8:32 AM

Reineke - Framed quiver moduli, cohomology, and quantum groups

Nakajima, Crawley-Boevey

Like before, attach each vertex a vector space M_i.

NEW: attach another vector space V_i (framing) to each vertex i, (V_i could be zero,)
and take $f_i: M_i \to V_i$.
(DO NOT require f_i to be injective.
This will make a compact moduli.)

Representation space becomes

$$R_{d,n} \coloneqq Rep_d \times \bigoplus_{i \in Q_0} \mathrm{Hom}_k(M_i, V_i)\,.$$

Quotient by GL_d:
$g \cdot (M, f) = \left(g \cdot M f_i(g_i^{-1} \cdot)\right)$
(Note that GL_d does not act on V_i.
That is, only vary M and f_i, but not V_i.)

Stable:
no non-zero subrep. $U \subset M$ contained in $Ker\ f$.
(In particular $Ker\ f$ itself is not a subrep.)

$M_{d,n} \coloneqq \{\text{stable framed rep.}\}/GL_d$.

Typical example:
$Gr_d(V)$.

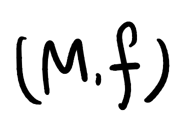
The following identifies with GIT,
which ensures the quotient is `good'.

De-framing:

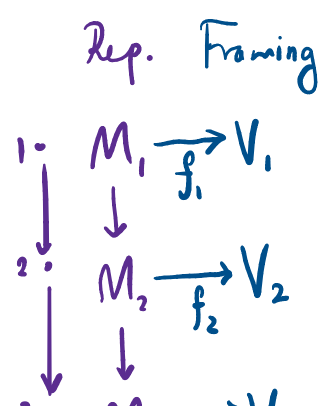
New quiver $\tilde{Q}$
with one more vertex ∞,
together with n_i arrows from i to ∞.
Put dim=1 over the vertex ∞.
Then take the character
$\Theta = -\infty^*$
for slope stability $\Theta(\alpha)/\Sigma\alpha$.
(The stability ineq. is taken in the common convention, which is reversed of King's convention.)

Prop.
$M^{ss}_{\tilde{d}}(\tilde{Q}) \cong M^{s}_{\tilde{d}}(\tilde{Q}) \cong M_{d,n}$.

Proof.
Obvious that rep. of $\tilde{Q}$ (with dim=1 over ∞)
corr. to rep. in $R_{d,n}$.
Slope of rep. $(M, \mathbb{C}_\infty)$ of $\tilde{Q}$ is
$-1/(1 + \Sigma \dim M)$.
For a proper subrep. of $(M, \mathbb{C}_\infty)$:
    If it is of the form $(N, \mathbb{C}_\infty)$,
    then $\Sigma \dim N < \Sigma \dim M$
    and hence satisfy the strict slope inequality.
    If it is $(N, 0)$,
    then it has $\Theta = 0$, and hence slope=0,
    which violates the slope inequality.
Hence GIT semi-stable iff GIT stable iff
does not have subrep. of the form $(N, 0)$.
This is equiv. to that
no non-zero subrep. $U \subset M$ contained in $Ker\ f$.

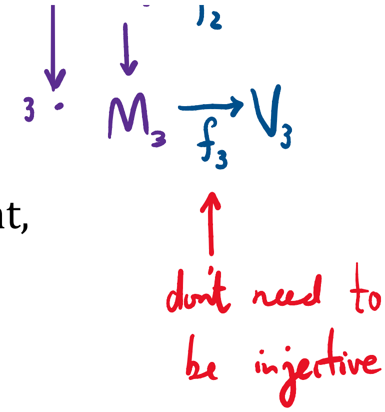


Prop.
Assume Q has no oriented cycle.
$M_{d,n}$ is projective.



Proof.
$\tilde{Q}$ does not have oriented cycle.
Then for each dim. vector,

Q does not have oriented cycle.
Then for each dim. vector,
the rep. with all arrows assigned to be 0
is the unique closed orbit.
Hence the GIT without character is just a point,
and hence the GIT quotient is projective.



Rmk.
Can take reversed framing

$$Rep_d \times \bigoplus_{i \in Q_0} \mathrm{Hom}_k(V_i, M_i)\,.$$

Stable means no proper subrep. U contains $\mathrm{Im}\, f$.
(Both ways are kind of minimality of the rep.)
The corresponding GIT character is
$\Theta = \infty^*$.

Aim:
Realize it as quiver Gr.,
describe it as iterated Gr. bundle,
and use it to compute cohomology ring.

Grassmannians of subrepresentations

Fix $X \in \mathrm{Rep}_d\, Q, e \leq d = \dim X$.
$Gr_e(X)$:

$$Gr_e(X) \subset \prod_i \mathrm{Gr}_{e_i}(X_i)$$

subset of elements that respect X.

Fix M_i.
$\mathrm{Gr}_{e_i}(X_i) = IHom(M_i, X_i)/GL(M_i)$.
$IHom_e(M, X)$:
the inverse image of $Gr_e(X)$.

Have natural map

$IHom_e(M,X) \to Rep_e$
by restriction of rep.

Want to regard (M,f) as subrep. of a fixed rep.
Problem:
f is not injective, and
the frame V_i is not a rep.

Idea: "enlarge" the target V_i.

Recall the projectives P_i and injectives $I_i = (e_i \cdot A)^*$.

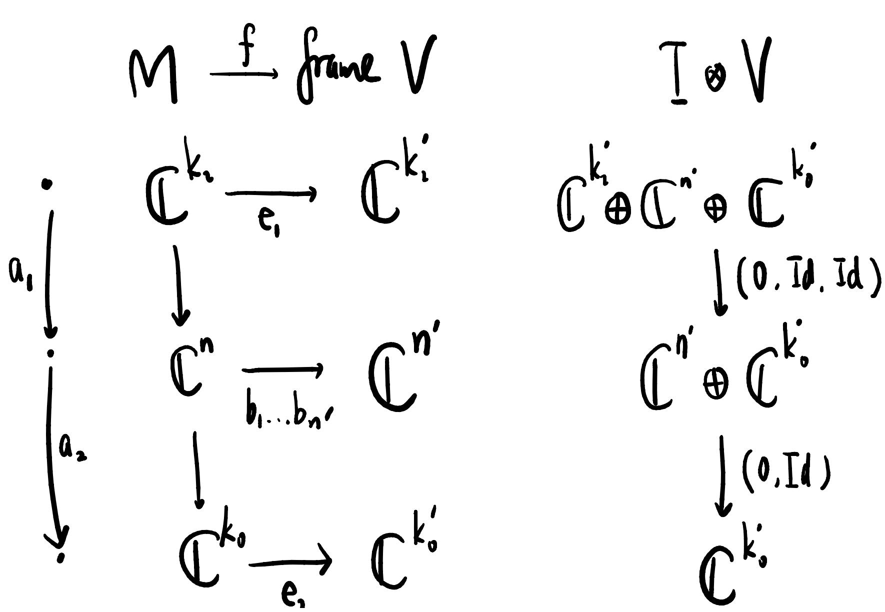
$$\mathrm{Hom}_Q(P_i, M) \simeq M_i \quad \text{and} \quad \mathrm{Hom}_Q(M, I_i) \simeq M_i^*$$

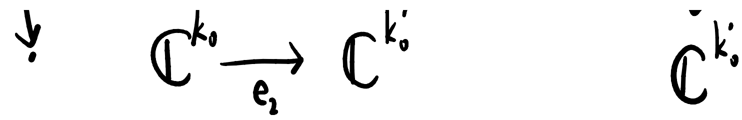
for any rep. M.

For the framing V,
Take $I \otimes V = \bigoplus I_i \otimes_{\mathbb{C}} V_i$.
(Think of several copies of I_i added together.)

$$(I \otimes V)_i \cong \bigoplus_{i \to j} V_j \, .$$

ex.





framed rep. → subrep. of $I \otimes V$:

Def. 3.6.

For $(M, f) \in R_{d,n}$,

define $\phi: M \to I \otimes V$ by:

First for sink i, $(I \otimes V)_i = V_i$. Set

$\phi_i = f_i: M_i \to V_i$.

For general vertex,

$$(I \otimes V)_i = \bigoplus_{i \to j} V_j = V_i \oplus \bigoplus_{a: t_a = i} \bigoplus_{h_a \to j} V_j .$$

$\phi_i = (f_i, \phi_{h(a)} \circ M_a)$

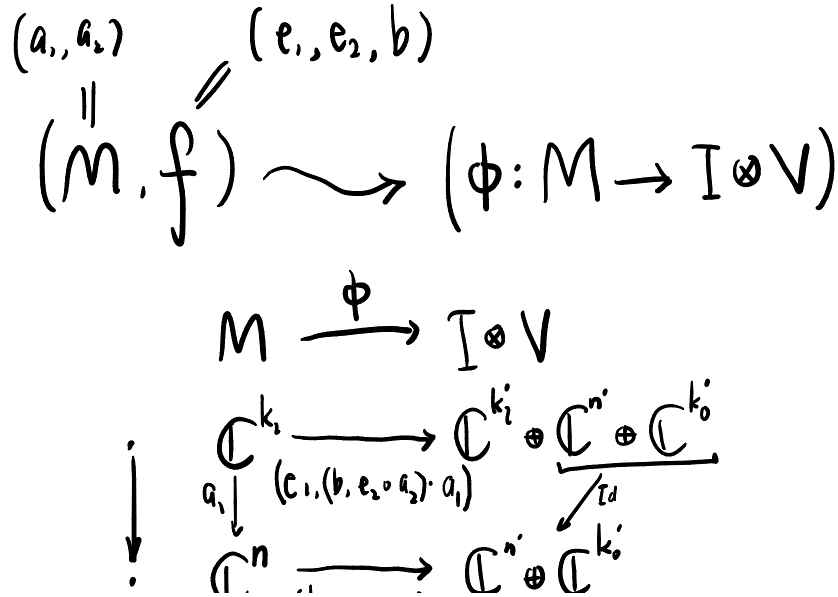
(This means: use M to follow the path, and apply f at every node of the path.)

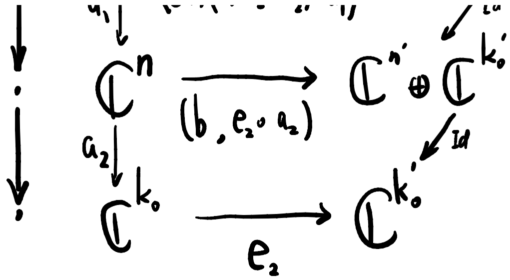
f is recovered from ϕ by projecting

$$\phi_i: M_i \to (I \otimes V)_i = \bigoplus_{i \to j} V_j$$

to V_i.

ex.





Lem. 3.7 and 3.8.
Ker ϕ is the max. subrep. of M contained in Ker f.
Hence (M, f) is stable iff ϕ is one-one.

Proof.
f_i is a component of ϕ, and so
$\phi_i(v) = 0 \Rightarrow f_i(v) = 0$.
Hence $Ker(\phi) \subset Ker(f)$.

If M' is subrep. contained in Ker f,
apply induction starting from sink to see
$M' \subset$ Ker ϕ.
Recall
$\phi_i = (f_i, \phi_{h(a)} \circ M_a)$
Since M' is subrep.,
$M_a(M'_{t(a)}) \subset M'_{h(a)}$
which lies in Ker ϕ by inductive hypothesis.

Prop. 3.9.
$M_{d,n} \cong Gr_d(I \otimes V)$.

Proof.
The above gives GL_d equiv. map from
stable points of $M_{d,n}$ to IHom$_d(I \otimes V)$.
The map is invertible by projecting ϕ_i to the first factor.

Relation between $Gr_d(I \otimes V)$ and $\prod_i Gr_{d_i}((I \otimes V)_i)$

Prop. 4.4.

$$\left(U_i \subset \bigoplus_{i\to j} V_j\right)_{i\in Q_0} \in \prod_i Gr_{d_i}\left((I \otimes V)_i\right)$$

lies in $Gr_d(I \otimes V)$

$$\Leftrightarrow U_i \subset V_i \oplus \bigoplus_{\alpha: t_\alpha = i} U_{h_\alpha} \quad \forall i.$$

Proof.

By Prop. 3.9, identify $Gr_d(I \otimes V)$ as $M_{d,n}$.

If $(U_i)_{i\in Q_0}$ lies in $Gr_d(I \otimes V)$,

$U_i = \mathrm{Im}\ \left(\phi_i = \left(f_i, \phi_{h(a)} \circ M_a\right)\right)$

for some $(M, f) \in M_{d,n}$.

By induction, the second component:

$\mathrm{Im}\ \phi_{h(a)} \circ M_a = \phi_{h(a)}(\mathrm{Im}\ M_a) \subset U_{h(\alpha)}$.

If $U_i \subset V_i \oplus \bigoplus_{\alpha: t_\alpha = i} U_{h_\alpha} \quad \forall i$,

want to produce (M, f).

choose isom. $M_i = \mathbb{C}^{d_i} \xrightarrow{\cong} U_i \subset (I \otimes V)_i$.

Projecting to the first factor V_i gives f.

Projecting to the second factor gives

$$M_i \to \bigoplus_{\alpha: t_\alpha = i} U_{h_\alpha} \cong \bigoplus_{\alpha: t_\alpha = i} M_{h(\alpha)}$$

and this defines the arrow map of M.

The above gives injective $\phi_i: M_i \to (I \otimes V)_i$

which is of the form $\left(f, \phi_{h(a)} \circ M_a\right)$

and hence (M, f) is stable.

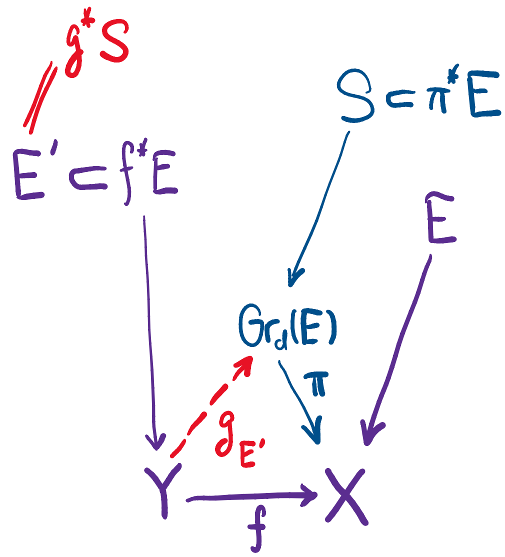
<u>Iterated Grassmannian bundle</u>

Given $p: E \to X$ v.b.

Have bundle $\pi: Gr_d(E) \to X$ and taut. bundle $S \subset \pi^* E$.

Universal property:

any sub-bundles of f^*E for any $f\colon Y \to X$ come from pulling back taut. S:



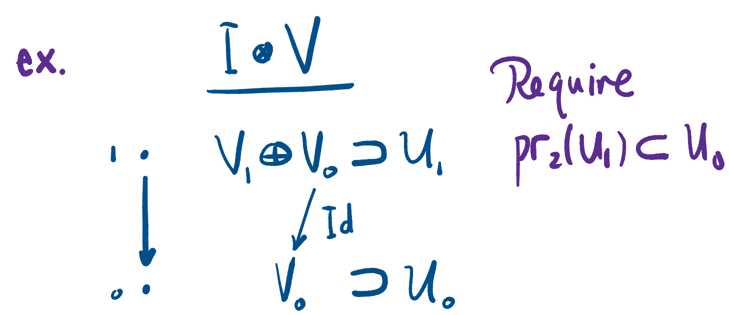
This is particularly simple if
$E \subset$ trivial bundle $V \times X$ for some V:
$Gr_d(E) \subset Gr_d(V) \times X$
consisting of (U, x) with $U \subset E_x$.
$S = \{(v, (U, x)) : v \in U \subset E_x\}$.

Universal quiver rep $\to M_{d,n}$
described using
$\boldsymbol{M_{d,n} \cong Gr_d(I \otimes V)}$:

<u>Univ. bundle of vertex i</u>:
subbundle $\boldsymbol{U_i} \subset$ trivial bundle $(I \otimes V)_i$ over $Gr_d(I \otimes V)$ consisting of $(v, (U_i)_i)\colon v \in U_i$.
(By Prop. 4.4,

$$\boldsymbol{U_i \subset V_i \oplus \bigoplus_{\alpha: t_\alpha = i} U_{h_\alpha}} \quad \forall i.$$

)



<u>Bundle morphism $\boldsymbol{U_i \to U_j}$ for arrows $\alpha\colon i \to j$</u>:
Projecting to second component $pr_\alpha(v) \in U_j$ for

$$v \in U_i \subset V_i \oplus \bigoplus_{\alpha: t_\alpha = i} U_{h_\alpha}.$$

Key construction to realize $M_{d,n}$ as iterated Gr bundle:

i_0: source vertex.
$\bar{Q}$: remove i_0 from Q.
Have $\bar{d}, \bar{n}$.

By Prop. 4.4, $(U_i)_i \in Gr_d(I \otimes V)$ satisfies

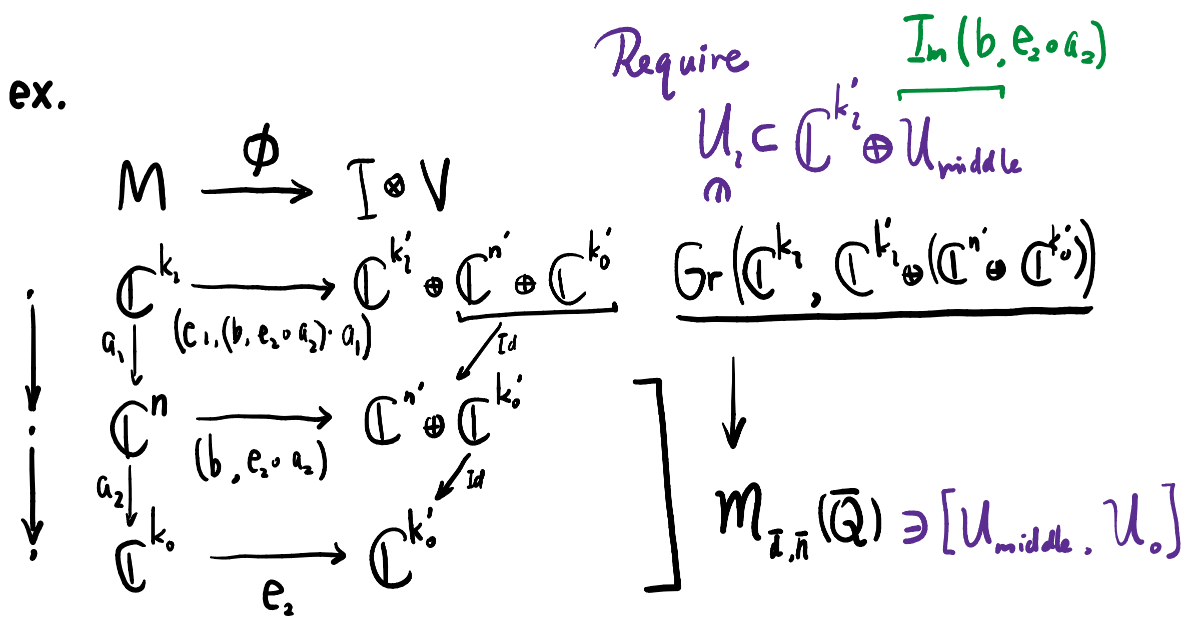
$$U_i \subset V_i \oplus \bigoplus_{\alpha : t_\alpha = i} U_{h_\alpha} \quad \forall i.$$

Thus $M_{d,n}(Q)$ is the subvar. in
the trivial $\mathrm{Gr}_{d_{i_0}}\left((I \otimes V)_{i_0}\right)$-bundle over $M_{\bar{d},\bar{n}}(\bar{Q})$
consisting of $\left(U_{i_0}, (U_i)_{i \in \bar{I}}\right)$ satisfying

$$U_{i_0} \subset V_{i_0} \oplus \bigoplus_{\alpha : t_\alpha = i_0} U_{h_\alpha} \subset (I \otimes V)_{i_0}.$$

The universal bundles agree under pullback of
$M_{d,n}(Q) \to M_{\bar{d},\bar{n}}(\bar{Q})$

ex.

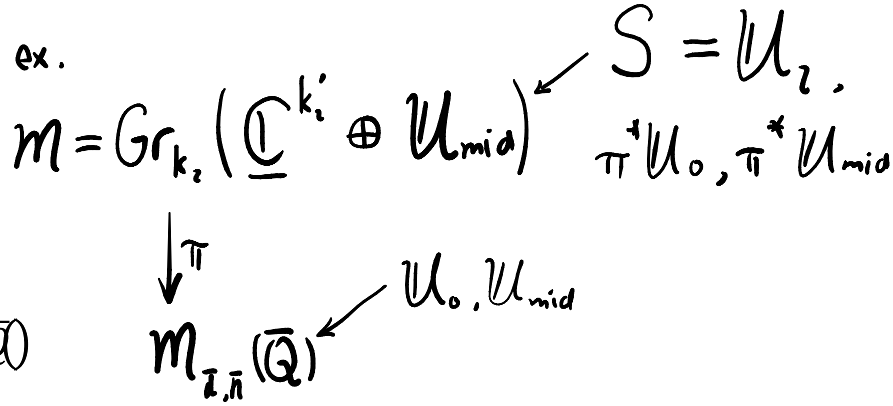


ex. $S = U_2$.

Lemma 4.9.
$M_{d,n}(Q) \to M_{\bar{d},\bar{n}}(\bar{Q})$
is isomorphic to the Gr. bundle

$$\mathrm{Gr}_{d_{i_0}}\left(\boldsymbol{V}_{i_0} \oplus \bigoplus_{a:i_0\to j} \boldsymbol{U}_j\right) \to M_{\bar{d},\bar{n}}(\bar{Q})$$

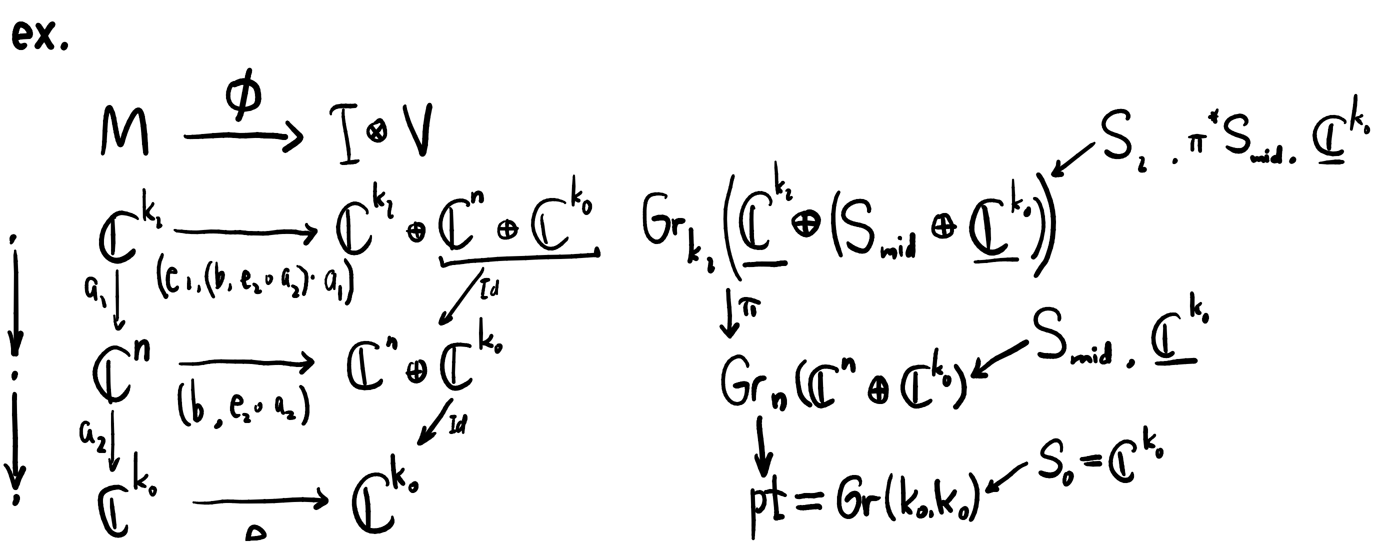
where $\boldsymbol{U}_j \to M_{\bar{d},\bar{n}}(\bar{Q})$ are univ. bundles
and $\boldsymbol{V}_{i_0}$ is trivial bundle.
The tautological bundle of this Gr. bundle is
the i_0-univ. bundle over $M_{d,n}(Q)$.

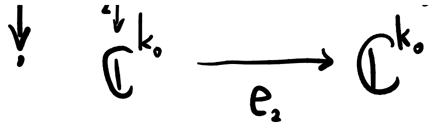


Repeat removing sources, get
Thm. 4.10.
$M_{d,n}$ is the total space M_1 of the iterated Gr bundles
$M_1 \xrightarrow{p_1} M_2 \xrightarrow{p_2} \dots \xrightarrow{p_{s-1}} M_s \xrightarrow{p_s} \mathrm{pt}$
where

$$M_i = \mathrm{Gr}_{d_i}\left(V_i \oplus \bigoplus_{i\to j} p_{i+1}^* \dots p_{j-1}^* S_j\right) \to M_{i+1}$$

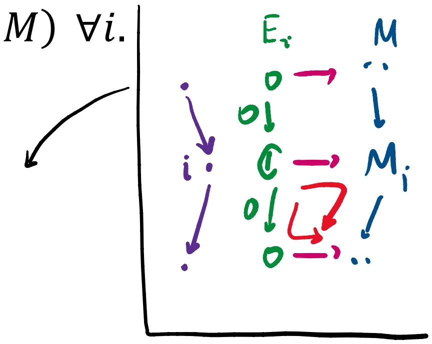
where S_j is the taut. bundle for Gr. bundle $M_j \to M_{j-1}$.
The universal bundles of $i \in Q_0$ over M_1 are
$p_1^* \dots p_{i-1}^* S_i$.





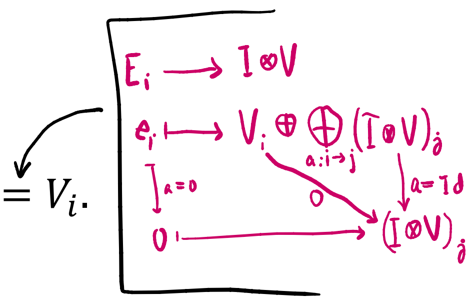
Remark about non-emptyness:

Lem. 4.1.
Given M,
$(M,f) \in R_{d,n}$ stable for some $f \Leftrightarrow n_i \geq \hom_Q(E_i, M)\ \forall i$.
(E_i denotes simple representation.)



($\mathrm{Hom}_Q(E_i, M) \cong \mathrm{Ker}\left(\bigoplus_{\alpha:i\to j} M_\alpha\right)$ by commutativity.
Note that when no outgoing arrows,
$\bigoplus_{\alpha:i\to j} M_\alpha = 0$ and hence $\mathrm{Ker}\left(\bigoplus_{\alpha:i\to j} M_\alpha\right) = M_i$.)

Proof.



$\to$)
Have injective morphism $\phi: M \to I \otimes V$.
Gives injective $\mathrm{Hom}_Q(E_i, M) \to \mathrm{Hom}_Q(E_i, I \otimes V) = V_i$.
Hence $\hom_Q(E_i, M) \leq n_i$.
$\leftarrow$)
Want to define $f: M_i \to V_i$.
To use the condition $n_i \geq \ker\left(\bigoplus_{\alpha:i\to j} M_\alpha\right)$
makes f injective on $\mathrm{Ker}\left(\bigoplus_{\alpha:i\to j} M_\alpha\right) \subset M_i$.

For each i, choose splitting
$$M_i = \mathrm{Ker}\left(\bigoplus_{\alpha:i\to j} M_\alpha\right) \oplus W_i \text{ and } V_i = \mathrm{Ker}\left(\bigoplus_{\alpha:i\to j} M_\alpha\right) \oplus V_i'.$$
(W_i may not form subrep.)
Define $f: M_i \to V_i$ by
projecting to $\mathrm{Ker}\left(\bigoplus_{\alpha:i\to j} M_\alpha\right)$ and inject to V_i.
Then $\mathrm{Ker}\, f = W$.

Stable: suppose $U \subset M$ is subrep. contained in Ker $f = W$.
$U = 0$:

At sink i, $U_i \subset W_i = 0$.
General i:
any arrow $i \to j$ maps W_i isom. to image
(since transverse to kernel).
Hence $U_i \subset W_i$ is isom. to image $U_a(U_i)$.
$U_a(U_i) \subset U_{h(a)}$ since U is subrep, and
$U_{h(a)} = 0$ by inductive hypothesis.
Hence $U_i = 0$.

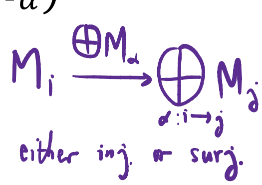
Prop. 4.3.
$\boldsymbol{M}_{d,n} \neq \emptyset$
$\Leftrightarrow n_i \geq \langle \vec{\imath}, d \rangle = \hom_Q(E_i, M) - \mathrm{ext}^1_Q(E_i, M)\ \forall i.$
($\vec{\imath}$ means the i-th basic vector.)
(Recall that

$$\langle \alpha, \beta \rangle = \sum_x \alpha(x)\beta(x) - \sum_a \alpha(t_a)\beta(h_a)\,.$$

$$\langle \vec{\imath}, d \rangle = d_i - \sum_{\alpha: i \to j} d_j\,.$$

)
Note that $\hom_Q(E_i, M)$ depends on the rep. M,
while $\langle \vec{\imath}, d \rangle$ does not.

Proof.
$\Rightarrow$ is obvious by Lem. 4.1.
$\leftarrow$)
Take M such that
rank$\left(\bigoplus_{\alpha: i \to j} M_\alpha\right)$ is max. $\forall i$.
Then $\hom_Q(E_i, M) = \ker\left(\bigoplus_{\alpha: i \to j} M_\alpha\right)$
$= \max(0, \langle \vec{\imath}, d \rangle)\,.$



Thus $n_i \geq \hom_Q(E_i, M)$.

Cohomology
Use the iterated Gr bundle structure to compute coho.

Chow ring:
$A^i(X)$: group of $\text{codim}_{\mathbb{C}} = i$ cycles up to rat. equivalence.
Formal Chern poly.
$P = \sum c_i t^i \in A^*(X)[|t|]$.

Given partition
$\lambda = (\lambda_1 \geq \cdots \geq \lambda_d \geq 0)$,
Schur poly.
$\Delta_\lambda(P) = \det\left(c_{\lambda_i+j-i}\right)_{i,j}$.

Thm. 5.1.
Consider
$\pi: Gr_d(E) \to X$
with taut. bundle S.

1. $A^i\left(Gr_d(E)\right) \cong \bigoplus_\lambda A^{i-|\lambda|}(X)$

   where $\lambda = (\text{rk}\, E - d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0)$,
   $|\lambda| = \sum \lambda_k$.

   RHS $\alpha_\lambda \in A^{i-|\lambda|}(X)$ corr. to
   $\sum_\lambda \Delta_\lambda(c_t(S)^{-1}) \cdot \pi^* \alpha_\lambda \in A^i\left(Gr_d(E)\right)$
2. $A^*\left(Gr_d(E)\right)$ is gen. by $A^*(X)$ together with $c_1(S), \ldots, c_d(S)$ with the relation that $c_t(S)^{-1}$ is a poly. of deg. at most $rk\ E - d$.
3. $A^* \to H^*$ is iso. for $Gr_d(E)$ iff it is for X.

4. $P_{Gr_d(E)}(q) := \sum_{i=0}^{\infty} \dim A^i(Gr_d(E))q^i$

equals to

$P_X(q) \cdot \binom{rk\, E}{d}$.

$\binom{n}{d} = \prod_{k=1}^{d} \frac{q^{n-d+k}-1}{q^k-1}$.

(Binomial coefficient.)

Apply the above to the iterated Gr. bundle for $M_{d,n}$.

Theorem 5.2. *For all $\mathcal{M}_{d,n} \neq 0$, the following holds:*

1. *The Chow ring $A^\bullet(\mathcal{M}_{d,n})$ has a linear basis*

$$\left\{ \prod_{i \in I} \Delta_{\lambda^i}\left(c_t(\mathcal{V}_i)^{-1}\right) \right\},$$

   indexed by tuples of partitions $(\lambda^i = (\lambda^i_1, \ldots, \lambda^i_{d_i}))_{i \in I}$ such that

$$n_i - \langle i, d \rangle \geqslant \lambda^i_1 \geqslant \cdots \geqslant \lambda^i_{d_i} \geqslant 0.$$

2. *As a ring, $A^\bullet(\mathcal{M}_{d,n})$ is generated by the Chern classes of the universal bundles $\mathcal{V}_i$ for $i \in I$, together with the following defining relations:*
   For each $i \in I$, the formal power series

$$\frac{\prod_{i \to j} c_t(\mathcal{V}_j)}{c_t(\mathcal{V}_i)}$$

   is a polynomial of degree at most $\leqslant n_i - \langle i, d \rangle$.

3. *Assume $k = \mathbf{C}$. Then the odd cohomology $H^{2\bullet+1}(\mathcal{M}_{d,n})$ vanishes, and the even cohomology $H^{2\bullet}(\mathcal{M}_{d,n})$ is isomorphic to the Chow ring $A^\bullet(\mathcal{M}_{d,n})$.*
4. *The Poincaré polynomial of $\mathcal{M}_{d,n}$ is given by*

$$\sum_{k=0}^{\infty} \dim A^k(\mathcal{M}_{d,n}) q^k = \prod_{i \in I} \begin{bmatrix} n_i + \sum_{i \to j} d_j \\ d_i \end{bmatrix}.$$

Have Schubert calculus, like

Pieri rule, Giambelli's formula, Littlewood-Richardson rule.

Group action on $M_{d,n}$

Have G_d-equiv. iso.
$R^s_{d,n} \xrightarrow{\cong} IHom_d(I \otimes V)$
from Prop. 3.9.
$Aut_Q(I \otimes V)$ on RHS commutes with G_d
and hence
$Aut_Q(I \otimes V)$ acts on $M_{d,n}$.

The action in terms of
$M_{d,n} \cong Gr_d(I \otimes V)$:
Note

$$\mathrm{End}_Q(I \otimes V) \simeq \bigoplus_{i,j} \mathrm{Hom}_Q(I_i \otimes V_i, I_j \otimes V_j) \simeq \bigoplus_{i \rightsquigarrow j} \mathrm{Hom}_k(V_i, V_j)$$

can be understood as lower triangular matrices (since Q has no oriented cycles).
$A_n := \mathrm{Aut}_Q(I \otimes V) \subset \mathrm{End}_Q(I \otimes V)$
consists of elements that have all ii-components to be invertible.
Moreover

$$A_n \subset \prod_{i \in I} GL((I \otimes V)_i)$$

$$R_d^{(n)} \xleftarrow{p} R^s_{d,n} \xrightarrow{\pi} \mathcal{M}_{d,n}$$

where $R_d^{(n)}$ is the image of $R^s_{d,n} \to R_d$
consisting of rep. M with $\hom_Q(E_i, M) \leq n_i \ \forall i$ (Lem. 4.1).
p is A_n-inv. and G_d-equiv.

Thm. 6.4.
$\pi p^{-1}(-)$ gives bijections between A_n-inv. subvar. of $M_{d,n}$ and G_d-inv. sub. of $R_d^{(n)}$.
Inclusions, closures, irreducibility and types of singularities are preserved.

Example. For the quiver $Q = i \to j \to k$, the group A_n is the subgroup of

$$\mathrm{GL}(V_1 \oplus V_2 \oplus V_3) \times \mathrm{GL}(V_2 \oplus V_3) \times \mathrm{GL}(V_3)$$

given by block matrices of the form

$$\left(\begin{bmatrix} a & & \\ b & d & \\ c & e & f \end{bmatrix}, \begin{bmatrix} d & \\ e & f \end{bmatrix}, [f] \right)$$

such that the matrices a, d and f are invertible.