Nakajima quiver variety

Saturday, February 1, 2020 11:46 AM

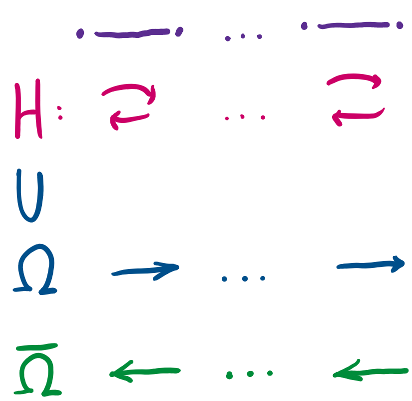
[Nakajima - Instantons on ALE spaces, quiver varieties and Kac-Moody Algebras]

- Provide nice examples of (non-compact) holomorphic symplectic varieties and resolutions of singularities
- Cohomologies give representations of Kac-Moody Lie algebras
- Naturally come up as mod. of ASD connections over $\widetilde{\mathbb{C}_2/\Gamma}$ [Kronheimer]

Hyper-Kaehler quotient

Given graph with no self edge.
H: set of all edges together with orientation.
Choose $\Omega \subset H$ with
$\Omega \cup \bar{\Omega} = H, \Omega \cap \bar{\Omega} = \emptyset$
such that Ω has no oriented cycle.

ex.



Fix Herm. v.s. V_k, W_k $\forall$ vertex k.
Dim. vector v, w.
Framed rep. space:

$$\mathbf{M} \stackrel{\text{def.}}{=} \left(\bigoplus_{h \in H} \operatorname{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}) \right) \oplus \left(\bigoplus_{k=1}^{n} \operatorname{Hom}(W_k, V_k) \oplus \operatorname{Hom}(V_k, W_k) \right).$$

W_k are the framing.

(We have started with M_Ω below.)

$(B, i, j) \in M$.

$\dim_{\mathbb{R}} \mathbf{M} = 2{}^t\mathbf{v}\mathbf{A}\mathbf{v} + 4{}^t\mathbf{v}\mathbf{w}$

where A is adj. matrix.

ex. Quiver with only one vertex.

frame

$$\begin{tikzcd} \cdot\, V \arrow[r, bend left, "j"] & W \arrow[l, bend left, "i"] \end{tikzcd}$$

Holomorphic symplectic:

$$\omega_{\mathbb{C}}((B, i, j), (B', i', j')) \stackrel{\text{def.}}{=} \sum_{h \in H} \operatorname{tr}(\varepsilon(h) B_h B'_{\bar{h}}) + \sum_{k=1}^{n} \operatorname{tr}(i_k j'_k - i'_k j_k)$$

where $\epsilon(h) = 1, -1$ for $h \in \Omega, \bar{\Omega}$ resp.

Decompose into Lagrangian subspaces:

$$\mathbf{M}_{\Omega} \stackrel{\text{def.}}{=} \left(\bigoplus_{h \in \Omega} \operatorname{Hom}(V_{\operatorname{out}(h)}, V_{\operatorname{in}(h)}) \right) \oplus \left(\bigoplus_{k=1}^{n} \operatorname{Hom}(W_k, V_k) \right)$$

$$\mathbf{M}_{\bar{\Omega}} \stackrel{\text{def.}}{=} \left(\bigoplus_{h \in \bar{\Omega}} \operatorname{Hom}(V_{\operatorname{out}(h)}, V_{\operatorname{in}(h)}) \right) \oplus \left(\bigoplus_{k=1}^{n} \operatorname{Hom}(V_k, W_k) \right).$$

$M = M_{\Omega} \oplus M_{\bar{\Omega}}$ can be understood as $T^* M_{\Omega}$.

M has Herm. metric induced from that of V, W.
HyperKaehler:
$J(m, m') = (-m'^{\dagger}, m^{\dagger}) \ \forall (m, m') \in M_{\Omega} \oplus M_{\bar{\Omega}} = M$
Where $(\square)^{\dagger}$ is the Herm. adj. for Hom space.

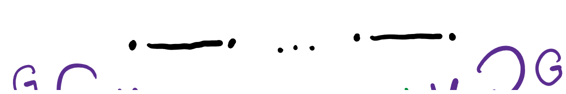
$$G = \prod_k U(V_k)$$

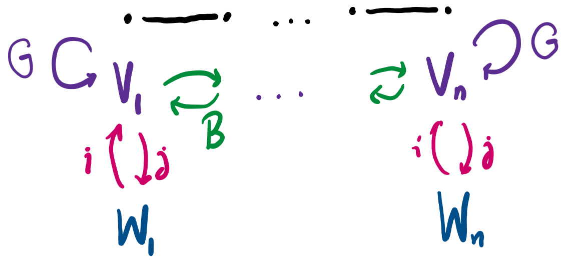
acts on M:

$$(B_h, i_k, j_k) \mapsto (g_{\operatorname{in}(h)} B_h g_{\operatorname{out}(h)}^{-1}, g_k i_k, j_k g_k^{-1})$$

preserving hyperKaehler structure.
(G does not act on the framing W.)

ex.





Moment maps of G and $G_{\mathbb{C}}$ (w.r.t. $\omega_{\mathbb{R}}$ and $\omega_{\mathbb{C}}$):

$$\mu_{\mathbb{R}}(B, i, j) = \frac{i}{2}\left(\sum_{h \in H : k = \mathrm{in}(h)} B_h B_h^{\dagger} - B_{\bar{h}}^{\dagger} B_{\bar{h}} + i_k i_k^{\dagger} - j_k^{\dagger} j_k\right)_k \in \bigoplus_k \mathfrak{u}(V_k) = \mathfrak{g}_{\mathbf{v}},$$

$$\mu_{\mathbb{C}}(B, i, j) = \left(\sum_{h \in H : k = \mathrm{in}(h)} \varepsilon(h) B_h B_{\bar{h}} + i_k j_k\right)_k \in \bigoplus_k \mathfrak{gl}(V_k) = \mathfrak{g}_{\mathbf{v}} \otimes \mathbb{C},$$

Fix $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in Z_v \oplus (Z_v \otimes \mathbb{C})$ where
$Z_v \subset \mathfrak{g}_v$ center ($i\mathbb{R} \cdot Id$ over each vertex k).

$Z_v \subset \mathbb{R}^{Q_0}$ (analog of real Cartan subalg.)
(Some $v(k)$ can be zero and so $\mathfrak{gl}_k = 0$.
Then Z_v may not be whole $\mathbb{R}^{Q_0}$.)

HK quotient:

$$\mathfrak{M}_\zeta = \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \overset{\text{def.}}{=} \{(B, i, j) \in \mathbf{M} \mid \mu(B, i, j) = -\zeta\}/G_{\mathbf{v}}.$$

$$\mathfrak{M}_\zeta^{\text{reg}} \overset{\text{def.}}{=} \{(B, i, j) \in \mu^{-1}(-\zeta) \mid \text{the stabilizer of } (B, i, j) \text{ in } G_{\mathbf{v}} \text{ is trivial}\}/G_{\mathbf{v}}.$$

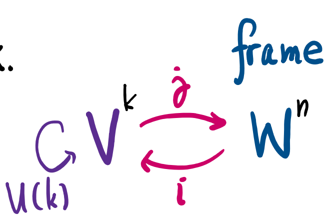
$$\dim_{\mathbb{R}} \mathbf{M} = 2\,{}^t\mathbf{v}\mathbf{A}\mathbf{v} + 4\,{}^t\mathbf{v}\mathbf{w}$$

${}^t v \cdot v$

$$\dim_{\mathbb{R}} \mathfrak{M}_\zeta^{\text{reg}} = \dim_{\mathbb{R}} M - 4 \dim_{\mathbb{R}} G_{\mathbf{v}} = 2\,{}^t\mathbf{v}(2\mathbf{w} - \mathbf{C}\mathbf{v})$$

where
$C = 2I - A$ and A is adj. matrix.
(Euler form)

ex.



$$\left.\left\{\begin{array}{ll} (1) & i \cdot j = -\underset{}{\overset{0 \,\|\, \text{set}}{\zeta_{\mathbb{C}}}} \cdot Id \\ (2) & i \cdot i^{\dagger} - j^{\dagger} j = -\underset{\text{set}}{\zeta_{\mathbb{R}}} \cdot Id \end{array}\right\}\right/ U(k) \qquad (i, j) \mapsto (g i, j g^{-1})$$

$U(k) \circlearrowright V \xleftarrow{i} W$

$\Big((2) \quad i \cdot i' - j' j = -\zeta_{\mathbb{R}} \cdot \mathrm{Id} \Big) \Big/ U(k) \qquad (i, j) \longmapsto (gi, jg')$

V set
0

$(2) \Rightarrow j$ inj. : If $0 \neq v \in \operatorname{Ker} j$, $v^{\dagger}(\mathrm{LHS})v = (v^{\dagger} \cdot i) \cdot (v^{\dagger} \cdot i)^{\dagger} > 0$,

but $v^{\dagger}(\mathrm{RHS})v < 0$.

Take $S = \operatorname{Im} j$.

$(1) \Rightarrow S \subset \operatorname{Ker} i$.

$A = ji$ maps W/S to S

(that is, $W \to S$ and $S \to 0$).

$\therefore \mathcal{M} \simeq T^{*}Gr(k, n) \ni (S, A)$.

ex. $k = 1$. $T^{*}\mathbb{P}^{n}$.

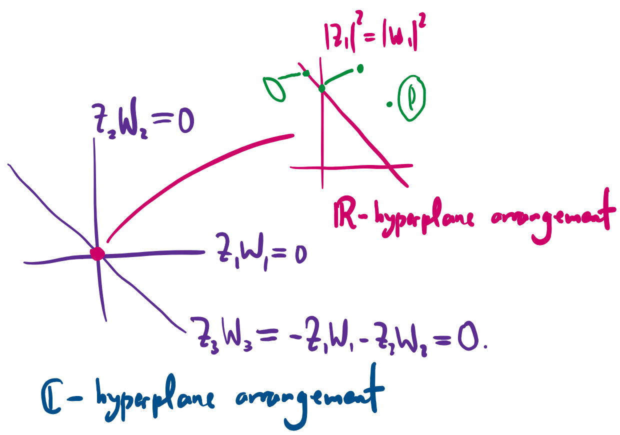
(1) $\sum_i w_i z_i = 0$.

(2) $\sum_i \left(|z_i|^2 - |w_i|^2 \right) = \zeta_{\mathbb{R}}$.

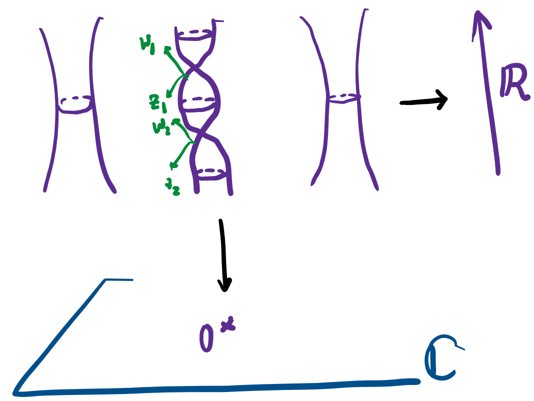
$\Big/ \quad (z, w) \longmapsto (\lambda^{-1} z, \lambda w)$

$\lambda \in U(1)$

$$\begin{array}{ccc} T^{*}\mathbb{P}^{n} & \xrightarrow{(|z_i|^2 - |w_i|^2)} & \mathbb{R}^{n} \\ \Big\downarrow {\scriptstyle (z_1 w_1 \ldots z_n w_n)} & & \\ \mathbb{C}^{n} & & \end{array}$$



ex. $\begin{tikzcd} T^*\mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{R}. \\ \mathbb{C} & \end{tikzcd}$



Smoothness

$R_+ := \{\theta \in \mathbb{Z}^n_{\geq 0} : \theta^t C\, \theta \leq 2\} - \{0\}.$
(Recall $\epsilon_x^t\, C\, \epsilon_x = 2$, and so positive real roots have
$\theta^t C\, \theta = 2$.)

$R_+(v) := \left\{\theta \in R_+ :\ \theta_k \leq v^{(k)}\ \forall k\right\}$ (finite set).

(wall)
$D_\theta := \{x \in \mathbb{R}^n : x \cdot \theta = 0\}$
where $\theta \in R_+$.

Thm. 2.8.
If

$$\zeta \in \mathbb{R}^3 \otimes \mathbb{R}^n \setminus \bigcup_{\theta \in R_+(\mathbf{v})} \mathbb{R}^3 \otimes D_\theta$$

then M_ζ is smooth.
(The HK quotient is complete.)

Proof.
Want: no non-trivial stabilizer.
Suppose has stabilizer: $(B, i, j) \in \mu^{-1}(-\zeta)$ is fixed by $g \in G_v$ (whose action is non-trivial).

Eigenspace decomp. of V_k by g_k:

$$V_k = \bigoplus_\lambda V_k(\lambda).$$

$g \in G \circlearrowright V$

$$V_k = \bigoplus_{\lambda} V_k(\lambda).$$

$g \in G \curvearrowright V$

$W \underset{j}{\overset{i}{\rightleftarrows}} V$

Recall action of g:

$$(B_h, i_k, j_k) \mapsto (g_{\mathrm{in}(h)} B_h g_{\mathrm{out}(h)}^{-1}, g_k i_k, j_k g_k^{-1})$$

Hence B preserves eigenspaces (with same eigenvalues);
$i(W) \subset V(1)$;
$j\big(V(\lambda)\big) = 0$ unless $\lambda = 1$.

Hence B_h restricted on $V_k(\lambda)$ (for $\lambda \neq 1$) defines a rep. in $M(\theta, 0)$ (where $\theta = \dim V(\lambda)$).
Momentum μ remains the same:

> μ is originally $-\zeta \cdot \mathrm{Id}$. Hence restricts to $-\zeta \cdot \mathrm{Id}$ on eigenspaces.

Consider the action of $G_\theta/U(1)$ on $G_\theta \cdot \{B_h\} \subset \mathrm{Rep}_\theta$.
If has non-trivial stabilizer, take eigen-decomp. of that.
Keep on doing this, until
$G_\theta/\mathrm{U}(1)$ acts freely on orbit of B_h.
Momentum of B_h is still $-\zeta \cdot \mathrm{Id}$.

Then B_h is a smooth point of $\mu^{-1}(-\zeta)/G_\theta$
(where μ here is $M(\theta, 0) \to Z_v \oplus (Z_v \otimes \mathbb{C})$).
Denote the corresponding rep. by V'.

Recall at a smooth point with framing,

$$\dim_{\mathbb{R}} \mathfrak{M}_\zeta^{\mathrm{reg}} = \dim_{\mathbb{R}} M - 4 \dim_{\mathbb{R}} G_{\mathbf{v}} = 2{}^t\mathbf{v}(2\mathbf{w} - \mathbf{C}\mathbf{v})$$

Without framing:
$\dim \mu^{-1}(-\zeta)/G_\theta = 2 - \theta^t C \theta \geq 0$
(where 2 comes from that $U(1)$ acts trivially in hyperKaehler quotient. $w = 0$.)
Hence the dim. vector of V': $\theta \in R_+(v)$.
By Lemma below,
$\zeta_{\mathbb{R}} \perp \theta$ and $\zeta_{\mathbb{C}} \perp \theta$, that is,
$\zeta \in \mathbb{R}^3 \otimes D_\theta$.
QED.

Lemma:

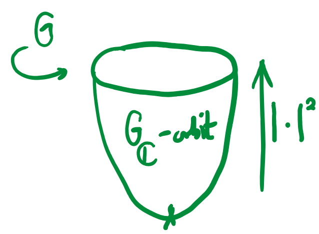
$\mu(V) = \zeta.$

Given any subrep. $V' \subset V$ with
$i(W) \subset (V')^\perp$ and $j(V') = 0$,
$\zeta_{\mathbb{R}} \perp \theta$ and $\zeta_{\mathbb{C}} \perp \theta$.

Proof.
Take
π: ortho. proj. $V \to V'$.
$i\pi \in \mathfrak{g}$ (skew-Herm.)
$e^{ti\pi} \in G$ fixes (B, i, j):
    $e^{ti\pi}$ is Id on $V'^{\perp}$ and $\in U(1)$ on V'.
    Hence acts trivially (as overall scaling) on V';
    i, j are only supported on $V'^{\perp}$.

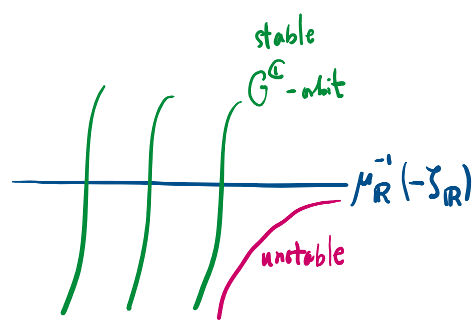
Hence Hamiltonian function in direction of $i\pi$ is constant,
which must be 0 since $\mu(B = 0, i = 0, j = 0) = 0$.
$\langle \mu(B, i, j), i\pi \rangle_{\mathfrak{g}} = 0$.
Since $\mu(B, i, j) = \zeta \cdot \mathrm{Id}$, get
$\langle \zeta \cdot \mathrm{Id}, i\pi \rangle_{\mathfrak{g}} = \sum_k \zeta_k \dim V'_k = 0 \in \mathbb{R} \oplus \mathbb{C}$.

<u>**Holomorphic description**</u>
[Kirwan],[Ness]
For $\xi_{\mathbb{R}} = 0$,
sympl = GIT quot:
$M_{(0,\xi_{\mathbb{C}})} \cong \mu_{\mathbb{C}}^{-1}(-\xi_{\mathbb{C}})//G_v^{\mathbb{C}}$
(affine GIT).
($G_v^{\mathbb{C}} = \prod_k GL(V_k)$.)



Key point: a $G^{\mathbb{C}}$-orbit is stable iff
the orbit has a minimum for $||^2$.
In this case,
each crit. pt. of $|\mu_{\mathbb{R}}|^2$ lies in $\mu_{\mathbb{R}}^{-1}\{0\}$.

For generic $(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$,
$\mathfrak{M}_{\zeta} \xrightarrow{\cong} H_{\zeta}^{s}/G_{\mathbf{v}}^{\mathbb{C}}$
where
$H_{\zeta}^{s} \overset{\text{def.}}{=} \{m \in \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}}) \mid \text{the } G_{\mathbf{v}}^{\mathbb{C}}\text{-orbit through } m \text{ intersects the level set } \mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}})\}$.



If $(0, \zeta_{\mathbb{C}})$ is generic (in the above sense that there is no
strictly semi-stable points),
then for $(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$,

all points are stable:
$H^s_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})} = \mu_{\mathbb{C}}^{-1}(-\xi_{\mathbb{C}})\ \forall \zeta_{\mathbb{R}}$.
Then complex structure is indep. of $\zeta_{\mathbb{R}}$.
(Only Kaehler structure depends.)
(No resolution occurs.)

The above def. of H^s_ζ is not practical enough.

Prop.
If $\zeta_{\mathbb{R}}^{(k)} > 0\ \forall k$,
$(B, i, j) \in H^s_\zeta$
$\Leftrightarrow$
No non-trivial subrep. of B lies in kernel of j_k,
namely,
for $(S_k \subset V_k)_k$, preserved by B and
$j_k(S_k) = 0\ \forall k$,
then $S_k = 0\ \forall k$.

<u>Resolution of singularity</u>

Denote
$\zeta = (0, \zeta_{\mathbb{C}})$ generic.
$\tilde{\zeta} = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$.
Have I-holomorphic
$\pi: \mathfrak{M}_{\tilde{\zeta}} \to \mathfrak{M}_\zeta$.

(Also known as affinization.
RHS is Spec of $G_{\mathbb{C}}$-inv. functions.)

Thm. 4.1.
1. π is proper.
2. $\pi^{-1}\left(M_\zeta^{\mathrm{reg}}\right) \cong M_\zeta^{\mathrm{reg}}$. ($\pi$ is resol. of sing.)
3. If $M_\zeta^{\mathrm{reg}} \neq \emptyset$, then $\pi^{-1}\left(M_\zeta^{\mathrm{reg}}\right)$ dense in $M_{\tilde{\zeta}}$.

<u>$\mathbb{C}^\times$-action</u>

Assume $(\zeta_{\mathbb{R}}, 0)$ generic.

$M_{\zeta_{\mathbb{R}}} := \cup_{\zeta_{\mathbb{C}}} M_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})} \to Z_{\mathbb{C}}$.
(Need to take union since $\mu_{\mathbb{C}} = \zeta_{\mathbb{C}}$ is generally not
preserved by the following action.
If we take $\zeta_{\mathbb{C}} = 0$, then preserved and don't need to take

union.)

$\mathbb{S}^1$-action:

$$(B_h, i_k, j_k) \mapsto (t^{(1-\varepsilon(h))/2} B_h, i_k, tj_k)$$

That is,
arrows in $M_{\bar{\Omega}}$ are multiplied by t,
those in M_Ω are unchanged.

Recall that

$$\mu_{\mathbb{R}}(B, i, j) = \frac{i}{2}\left(\sum_{h \in H: k=\mathrm{in}(h)} B_h B_h^\dagger - B_{\bar{h}}^\dagger B_{\bar{h}} + i_k i_k^\dagger - j_k^\dagger j_k\right)_k \in \bigoplus_k \mathfrak{u}(V_k) = \mathfrak{g}_{\mathbf{v}},$$

$$\mu_{\mathbb{C}}(B, i, j) = \left(\sum_{h \in H: k=\mathrm{in}(h)} \varepsilon(h) B_h B_{\bar{h}} + i_k j_k\right)_k \in \bigoplus_k \mathfrak{gl}(V_k) = \mathfrak{g}_{\mathbf{v}} \otimes \mathbb{C},$$

$\mu_{\mathbb{R}} = -\zeta_{\mathbb{R}}$ preserved;
(Note: this is not preserved if we take $t \in \mathbb{C}^\times$.)
$\mu_{\mathbb{C}}\big(t \cdot (B, i, j)\big) = t\mu_{\mathbb{C}}(B, i, j)$.

Use this action (and its moment map flow) to understand topology.

THEOREM 5.1. *The S^1-action on $\mathcal{M}_{\zeta_{\mathbb{R}}}$ has the following properties:*

(1) *The natural projection map $\mathcal{M}_{\zeta_{\mathbb{R}}} \to Z \otimes \mathbb{C}$ is equivariant; here we make S^1 act on the vector space $Z \otimes \mathbb{C}$ with weight 1. (In particular, $\mathfrak{M}_{(\zeta_{\mathbb{R}}, 0)}$ admits an S^1-action.)*

(2) *It preserves the complex structure I and the metric.*

(3) *The holomorphic symplectic form $\omega_{\mathbb{C}}$ transforms as $\omega_{\mathbb{C}} \to t\omega_{\mathbb{C}}$.*

(4) *The corresponding moment map*

$$F([B, i, j]) = \sum_{h \in \Omega} \|B_h\|^2 + \sum_k \|j_k\|^2$$

is proper.

(5) *The action is extended to a holomorphic (with respect to I) $\mathbb{C}^*$-action. If we use the holomorphic description $\mathfrak{M}_{(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})} = H^{\mathrm{s}}_{(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})}/G_{\mathbf{v}}^{\mathbb{C}}$, the $\mathbb{C}^*$-action is given by*

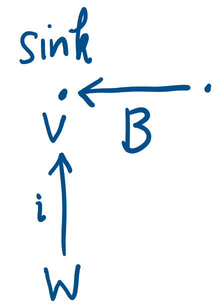
$$G_{\mathbf{v}}^{\mathbb{C}}(B_h, i_k, j_k) \mapsto G_{\mathbf{v}}^{\mathbb{C}}(t^{(1-\varepsilon(h))/2} B_h, i_k, tj_k).$$

Recall

$$\omega_{\mathbb{C}}((B, i, j), (B', i', j')) \stackrel{\text{def.}}{=} \sum_{h \in H} \mathrm{tr}(\varepsilon(h) B_h B'_{\bar{h}}) + \sum_{k=1}^n \mathrm{tr}(i_k j'_k - i'_k j_k)$$

(4):
Assume not. Then there exists a sequence $[(B, i, j)_l]$ in

$M_{\zeta_{\mathbb{R}}}$ that has no convergent subsequence, whose F-image lies in a bounded interval.
$|(B,i,j)_l| \to \infty$ or otherwise must has convergent subseq.
Take $(B,i,j)_l/|(B,i,j)_l|$ lying in the unit sphere and hence must have convergent subseq.
$\mu_{\mathbb{R}}\big((B,i,j)_l/|(B,i,j)_l|\big) \to 0$ (since $\mu_{\mathbb{R}}\big((B,i,j)_l\big) = \zeta_{\mathbb{R}}$)
and
$F\big((B,i,j)_l/|(B,i,j)_l|\big) \to 0$ (since $F\big((B,i,j)_l\big)$ is bounded).
Then the limit (B,i,j) has
$|(B,i,j)| = 1, \mu_{\mathbb{R}}(B,i,j) = 0$ and $F(B,i,j) = 0$.
$F = 0$ means (B,i,j) is usual framed quiver rep. (without doubling).
$\mu_{\mathbb{R}} = 0$ implies at sink, $i = 0$ and arrow maps to sink $= 0$.
Inductively, $(B,i,j) = 0$. But $|(B,i,j)| = 1$!



In (5), still need to take $M_{\zeta_{\mathbb{R}}} := \cup_{\zeta_{\mathbb{C}}} M_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})}$ since $\zeta_{\mathbb{C}} \mapsto t\zeta_{\mathbb{C}}$.
The holo. description implies the new point has orbit closure intersects the orbit closure of $\left(t^{1-\frac{\epsilon(h)}{2}} B_{h},, i_k, tj_k\right)$.
Want to argue it has the same orbit as $\left(t^{1-\frac{\epsilon(h)}{2}} B_{h},, i_k, tj_k\right)$.

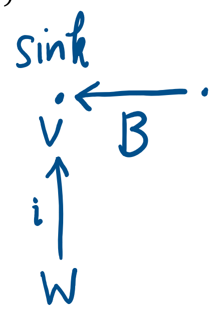
Since the moment map F has compact fibers,
the corresponding Ham. flow
(which is J of the $\mathbb{S}^1$-flow)
is complete.
The new point after action lies in $M_{(\zeta_{\mathbb{R}},t\zeta_{\mathbb{C}})}$, whose representative has $G_{\mathbb{C}}$-orbit of max. dim.
(All points are strictly stable since $\zeta_{\mathbb{R}}$ is generic.)
Hence it lies in $G_{\mathbb{C}} \cdot \left(t^{1-\frac{\epsilon(h)}{2}} B_{h},, i_k, tj_k\right)$.
(Note that $\left(t^{1-\frac{\epsilon(h)}{2}} B_{h},, i_k, tj_k\right) \notin \mu_{\mathbb{R}}^{-1}\{\zeta_{\mathbb{R}}\}$!)

The $\mathbb{C}^\times$-action preserve $\cup_{\zeta_{\mathbb{C}}} H^s_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})}$, since subrepresentations survive under scaling of arrows.

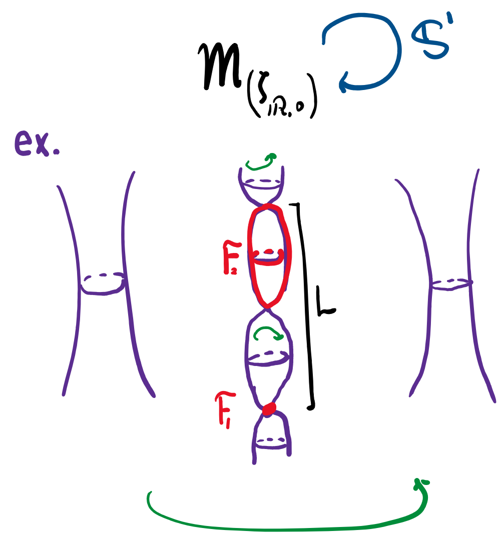
From the holomorphic description,
Thm. 5.2.
$\pi: M_{\zeta_{\mathbb{R}}} \to M_0 := \cup_{\zeta_{\mathbb{C}}} M_{(0,\zeta_{\mathbb{C}})}$ is $\mathbb{C}^\times$-equiv.
Note that in this notation,
M_0 is DIFFERENT from $M_{(0,0)}$!

$\pi: M_{\zeta_{\mathbb{R}}} \to M_0$
can be understood as simultaneous resolution
(of every fiber of $M_0 := \cup_{\zeta_{\mathbb{C}}} M_{(0,\zeta_{\mathbb{C}})} \to Z_{\mathbb{C}}$).

[**Slodowy**]:
$M_{(\zeta_{\mathbb{R}},0)}$ is homotopy equiv. to $L := \pi^{-1}\{0 \in M_{(0,0)}\}$.
(Contraction of $M_{(0,0)}$ to 0 induces contraction of $M_{(\zeta_{\mathbb{R}},0)}$ to L.)



$\mathbb{C}^{\times}$-fixed point set F must be contained in $M_{(\zeta_{\mathbb{R}},0)}$
(since the base point $\zeta_{\mathbb{C}}$ needs to be fixed by $\mathbb{C}^{\times}$).
Since π is equiv, $\pi(F)$ must be contained in
fixed point set of $M_{(0,0)}$.
Fixed point set of $M_{(0,0)}$ is just [0]:
(B, i, j) fixed by $\mathbb{C}^{\times}$ implies
it has doubled part $= 0$.
$\zeta_{\mathbb{R}} = 0$ implies $(B, i, j) = 0$ inductively
like proof of (4) above .



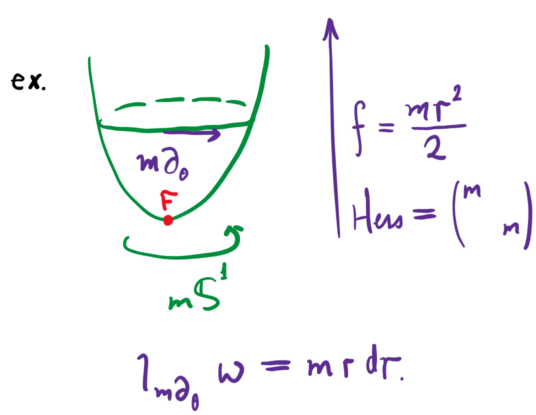
Thus
$F \subset \pi^{-1}\{0\} =: L$.

Homology

Denote conn. cpnt. of F by $F_1 \dots F_p$.
f: moment map of the $\mathbb{S}^1$-action on $M_{(\zeta_{\mathbb{R}},0)}$.
Perfect Morse function.
Proper by Thm. 5.1 (4).
grad $f = 2I \cdot V$.

F is the set of critical points of f.
$\mathbb{S}^1$-action on $T_x M_{(\zeta_{\mathbb{R}},0)}$ for $x \in F$
gives weight decomp.

$$T_x M_{(\zeta_{\mathbb{R}},0)} = \bigoplus_m H^m .$$



$T_x F_\alpha = H^{m=0}$.
$\mathrm{Hess}_x(f)$ acts on H^m as multi. by m.

$$\mathrm{Ind}(x) = m_\alpha \coloneqq \sum_{m<0} \dim_{\mathbb{R}} H^m \,.$$

(Points in the same fixed component have the same index.)

Since $M_{(\zeta_{\mathbb{R}},0)}$ is homotopy equiv. to L,

Prop. 5.7.

$$H_i\big(M_{(\zeta_{\mathbb{R}},0)}\big) \cong H_i(L) \cong \bigoplus_{1\le\alpha\le p} H_{i-m_\alpha}(F_\alpha)\,.$$

The second equality is due to f being perfect Morse.

In general,

$\mathrm{Ind}(x) = \dim_{\mathbb{R}} M - \dim_{\mathbb{R}} F_\alpha - \#(\text{positive directions})$.

We can do better in this case:

Lem. 5.6.

For each α (indexing of fix set),

$m_\alpha = \dim_{\mathbb{C}} M_{(\zeta_{\mathbb{R}},0)} - \dim_{\mathbb{R}} F_\alpha$.

Proof.

Consider $\omega_{\mathbb{C}}\colon T_x M_{(\zeta_{\mathbb{R}},0)} \wedge T_x M_{(\zeta_{\mathbb{R}},0)} \to \mathbb{C}$.

$\omega_{\mathbb{C}}$ becomes $t\omega_{\mathbb{C}}$ under the action.

Hence it has weight 1.

$\omega_{\mathbb{C}}\colon H^m \cong H^{1-m}$.

(Recall that H^m are the weight spaces of $T_x M_{(\zeta_{\mathbb{R}},0)}$.)

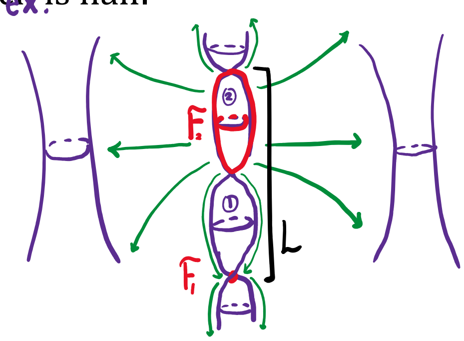
Then

$$\sum_{m\le 0} \dim H^m = \sum_{m>0} \dim H^m\,.$$

They add up to $\dim_{\mathbb{R}} M_{(\zeta_{\mathbb{R}},0)}$, and hence each is half.

LHS $= m_\alpha + dim_{\mathbb{R}} F_\alpha$.

ex.



$m_1 = 2.$
$m_2 = 0.$

Thm. 5.8.

$L = \pi^{-1}(0)$ decomposes into

{points that flows to F_α}

for $\alpha = 1, \dots, p$.

Closure of each is an irred. component.

L is Lag. w.r.t. $\omega_{\mathbb{C}}$.

Reason:

ω becomes $t\omega$ under pull-back by the $\mathbb{C}^\times$-action.

Thus $\omega(t \cdot u, t \cdot v) = t\omega(u, v)$.
If $u, v \in TL$, LHS is bounded for $t \to \infty$.
(Paths are gradually sent to the fixed locus by the action.)
Hence $\omega(u, v) = 0$,
L must be isotropic.
Also it is half dim. since at fixed point,

$$T_x L = \bigoplus_{m \leq 0} H^m .$$

ex. Cotangent bundle of A_n flag
$F(v_1, \dots, v_n; r)$ for
$r > v_1 > \cdots > v_n > 0$.
$T^*F = \{(\phi, A) \in F \times \mathrm{End}(\mathbb{C}^r) : A(E^k) \subset E^{k+1}\}$
where $\phi = (E^0 = \mathbb{C}^r \supset E^1 \supset \cdots \supset E^{n+1} = 0)$.

Take $M_\zeta(v, w)$ where
$v = (v_1, \dots, v_n), w = (r, 0, \dots, 0)$.

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[r] \arrow[d, phantom, "V"] & \overset{2}{\bullet} \arrow[r] & \overset{3}{\bullet} \arrow[r] & \cdots \arrow[r] & \overset{n-2}{\bullet} \arrow[r] & \overset{n-1}{\bullet} \arrow[r] & \overset{n}{\bullet} \\
V \arrow[d, shift right, "j_1"'] & & & & & & \\
\mathbb{C}^r \arrow[u, shift right, "i_1"'] & & & & & &
\end{tikzcd}$$

Thm. 7.3.
Let $\zeta_{\mathbb{R}} = \left(\frac{i\zeta_{\mathbb{R}}^{(1)}}{2}, \dots, \frac{i\zeta_{\mathbb{R}}^{(n)}}{2}\right)$
where $\zeta_{\mathbb{R}}^{(k)} > 0, k = 1, \dots, n$.

$M_{(\zeta_{\mathbb{R}}, 0)} \cong T^*F$.

Proof.
$B_{k-1,k}$ (arrow from k to $k-1$) and j_1 are injective:
the last vertex:
$$B_{n,n-1}B_{n,n-1}^{\dagger} - B_{n-1,n}^{\dagger}B_{n-1,n} = -\zeta_{\mathbb{R}}^{(n)} 1_{V_n}$$
hence $B_{n-1,n}$ is injective.
Suppose $B_{k-1,k}$ is inj.
At the vertex $(k-1)$:
$$B_{k-1,k}B_{k,k-1} - B_{k-1,k-2}B_{k-2,k-1} = 0$$
Hence $Ker\ B_{k-2,k-1} \subset Ker\ B_{k,k-1}$.

$$B_{k-1,k}B^{\dagger}_{k-1,k} - B^{\dagger}_{k,k-1}B_{k,k-1} + B_{k-2,k-1}B^{\dagger}_{k-1,k-2} - B^{\dagger}_{k-1,k-2}B_{k-2,k-1}$$
$$= -\zeta_{\mathbb{R}}^{(k-1)}1_{V_{k-1}}$$

$V_1 \hookrightarrow V_2 \subset V_3 \dots \subset V_n$

Hence $B_{k-2,k-1}$ is inj.

Similarly, consider the equations at the first vertex,
get j_1 inj.

Thus
$M_{(\zeta_{\mathbb{R}},0)} \subset \{(B,i,j): \mu_{\mathbb{C}}(B,i,j) = 0, B_{k-1,k} \text{ and } j_1 \text{ are inj.}\}/G_v^{\mathbb{C}}$.

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \longrightarrow \cdots \longrightarrow \overset{n-2}{\bullet} \longrightarrow \overset{n-1}{\bullet} \longrightarrow \overset{n}{\bullet}$$

From RHS to T^*F:
Get a flag $E^k = \operatorname{Im} j_1 B_{1,2} \dots B_{k-1,k}$.
$E^1 \supset \dots \supset E^n$.
Take $A = j_1 i_1 \in \operatorname{End} \mathbb{C}^r$.
$A(E^k) \subset E^{k+1}$:
$j_1 i_1 j_1 B_{1,2} \dots B_{k-1,k} = j_1 B_{1,2} B_{2,1} B_{1,2} \dots B_{k-1,k}$
(like moving $B_{1,2}$ to the left and create $B_{2,1}B_{1,2}$)
$= j_1 B_{1,2} \dots B_{k-1,k} B_{k,k-1} B_{k-1,k} = j_1 B_{1,2} \dots B_{k-1,k} B_{k,k+1} B_{k+1,k}$
and hence image lies in $Im(j_1 B_{1,2} \dots B_{k-1,k} B_{k,k+1}) = E^{k+1}$.

Surjective:
construct $j_1, B_{k-1,k}$ according to the flag E.
$A(\mathbb{C}^r) \subset E^1 = \operatorname{Im} j_1 \overset{j_1}{\cong} V_1$ defines i_1.
Similarly $A(E^k) \subset E^{k+1}$ defines $B_{k+1,k}$.

Injective:
If (B,i,j) and (B',i',j') give the same (E,A),
can make $B_{k-1,k} = B'_{k-1,k}$ and $j = j'$ by $G_v^{\mathbb{C}}$.
$A = j_1 i_1 = j'_1 i'_1 = j_1 i'_1$
implies $i_1 = i'_1$ since j_1 is inj.
$B_{1,2}B_{2,1} + i_1 j_1 = 0 = B'_{1,2}B'_{2,1} + i'_1 j'_1$
and $B_{1,2} = B'_{1,2}$ implies $B_{2,1} = B'_{2,1}$.
Keep on doing this, get $(B,i,j) = (B',i',j')$.

$M_{(\zeta_{\mathbb{R}},0)} \to T^*F$
is open and closed, and hence is iso:
Whether $G_v^{\mathbb{C}}$-orbit has solution to $\mu_{\mathbb{R}} = \zeta_{\mathbb{R}}$ is open cond.

The map $A = j_1 i_1 \colon M_{(\zeta_{\mathbb{R}},0)} \to \mathrm{End}(\mathbb{C}^r)$ is proper and hence closed.
([Kraft-Procesi] proved $M_0 \to \mathrm{End}(\mathbb{C}^r)$ is isom. to closure of conj. class of a nilpotent matrix.)