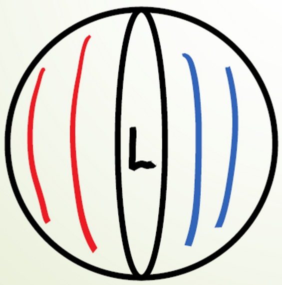
Quiver algebras as noncommutative mirrors

Monday, April 27, 2020 2:05 PM

Cho; Hong; Lau - Noncommutative homological mirror functor

Mirror symmetry: duality between symplectic and complex geometries

Symplectic geometry	Complex geometry
Symplectic form ω	Calabi-Yau volume form Ω
Lagrangian submanifolds	Holomorphic vector bundles
Gromov-Witten invariants	Integrals $\int \Omega$
$H^{p,q}$	$H^{q,p}$

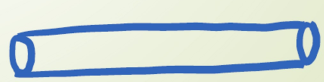


$$\frac{1}{4\pi i}\oint \left(z+\frac{A}{z}\right)^2 \frac{dz}{z} = A$$

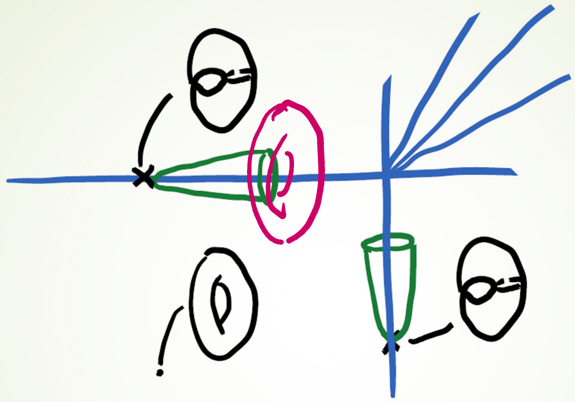
Left and right hemispheres bounded by the equator L add up to area A.

Strominger-Yau-Zaslow: Mirror symmetry is T-duality

- **Fundamental geometric guiding principle**.
- O_p in $\check{M}$ is mirror to a Lagrangian brane L in M.
  O_p has n dimensional deformation space. Thus expect $h^1(L) = n$.
  $Ext^*(O_p, O_p) = \Lambda^* \mathbb{C}^n$. So $L \cong T^n$ cohomologically.
- O_p moves in the whole space.
  Thus L should be a leaf of a foliation, or simply a (special) **Lagrangian fibration**.
- Need to complexify. Take (L, ∇).
  ∇ is a flat $U(1)$ connection in $Hom(\pi_1(L), U(1)) = (T^n)^*$.
- $\{\text{flat } U(1) \text{ connections on } L\} \cong T^*$ gives a torus in $\check{M}$.
  Thus $\check{M}$ should admit a **dual torus fibration**.

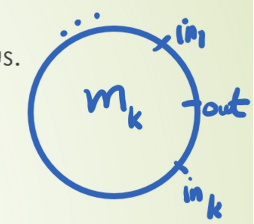


Quantum corrections from **singular fibers**



- Singular fibers occur. Construct and glue in their "dual" to complete the SYZ mirror. [**Hong-L.-Kim, Ekholm-Rizell-Tonkonog**]
  Glue up a mirror functor. [**Cho-Hong-L.**]
- Holomorphic discs from singular fibers ⇒ wall-crossing and gluing of dual tori. [**Auroux, Chan-L.-Leung, Auroux-Abouzaid-Katzarkov, Seidel, Pascaleff-Tonkonog**]
- Interacts and produce scattering diagram. [**Kontsevich-Soibelman, Gross-Siebert. Analytic approach: Chan-Leung-Ma. Fukaya, Tu, Abouzaid: family Floer theory for tori. Lin: K3 surfaces.**]

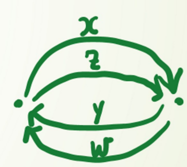
Quantum corrections from A-side



- Need $\boldsymbol{HF}^*(\boldsymbol{L},\boldsymbol{L}) \cong H^*(T)$ instead of $H^*(L)$. L does not need to be a torus. [**Fukaya-Oh-Ohta-Ono toric. Cho-Hong-L. immersed Lagrangians**].
- Obstruction term:

$$m_0^L = W \cdot [L] + \sum h_Y \cdot Y \in C^*(L).$$

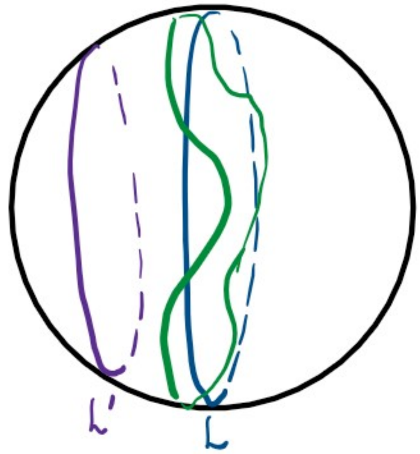
- **Landau-Ginzburg model** $(\check{M} \coloneqq \{L\colon h_Y = 0 \ \forall\, Y\}, W)$.
- May have no commutative solutions to $h_Y = 0$.
  Noncommutative mirror construction [**Cho-Hong-L.**].
  Ex. Noncommutative resolution of conifold.



- Need **Novikov field**

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{A_i} \colon A_i \to +\infty \right\}.$$

Mirror $\check{M}$ defined over Λ.



The equator L is Hamiltonian non-displaceable, while L' is not.
The usual cohomology of L does not tell this.
Need Lagrangian Floer cohomology.

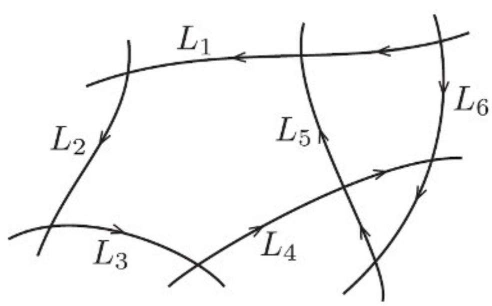
$\mathbb{L} = \{L_1, \dots, L_k\}$.

DEFINITION 6.1. $Q = Q^{\mathbb{L}}$ is defined to be the following graph. Each vertex v_i of Q corresponds to a Lagrangian $L_i \in \mathbb{L}$. Thus the vertex set is

$$Q_0^{\mathbb{L}} = \{v_1, \cdots, v_k\}.$$

Each arrow from v_i to v_j corresponds to odd-degree Floer generator in $\mathrm{CF}^{\bullet}(L_i, L_j)$ (which is an intersection point between L_i and L_j). In particular for $i = j$, we have loops at v_i corresponding to the odd-degree immersed generators of L_i.

Q will be sometimes called the *endomorphism quiver* of $\mathbb{L}$.



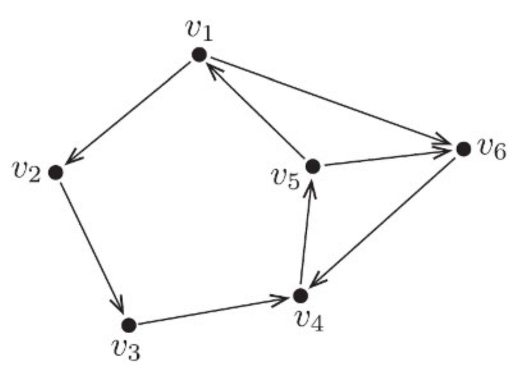


FIGURE 5. Endomorphism quiver $Q^{\mathbb{L}}$

The construction is briefly summarized as follows. The odd-degree Floer-theoretical endomorphisms of $\mathbb{L}$ are described by a directed graph Q (so-called a quiver). The path algebra ΛQ is regarded as the noncommutative space of formal deformations of $\mathbb{L}$. Each edge e of Q corresponds to an odd-degree Floer generator X_e and a formal dual variable $x_e \in \Lambda Q$. Consider the formal deformation $b = \sum_e x_e X_e$. The obstruction is given by

$$m_0^b = m(e^b) = \sum_{k \geq 0} m_k(b, \ldots, b).$$

A novel point is *extending the notion of weakly unobstructedness by Fukaya-Oh-Ohta-Ono* [**FOOO09**] *to the noncommutative setting*. The corresponding weak Maurer-Cartan equation is

$$m_0^b = \sum_{i=1}^{k} W_i(b) \mathbf{1}_{L_i}$$

where $\mathbf{1}_{L_i}$ is the Floer-theoretical unit corresponding to the fundamental class of L_i (we assume that Fukaya A_∞-category is unital). The solution space is given by a quiver algebra with relations $\mathscr{A} = \Lambda Q / R$ where R is the two-sided ideal generated by weakly unobstructed relations. The end product is a noncommutative Landau-Ginzburg model

$$\left(\mathscr{A}, W = \sum_i W_i \right).$$

We call this to be a generalized mirror of X, in the sense that there exists a natural functor from the Fukaya category of X to the category of (noncommutative) matrix factorizations of $(\mathscr{A}, W)$. It is said to be 'generalized' in two reasons. First, the construction can be regarded as a generalization of the SYZ program where we replace Lagrangian tori by immersions. Second, the functor needs not to be an equivalence, and so $(\mathscr{A}, W)$ needs not to be a mirror of X in the original sense.

THEOREM 1.1 (Theorem 4.7). *There exists an A_∞-functor $\mathscr{F}^{\mathbb{L}} : \mathrm{Fuk}(X) \to \mathrm{MF}(\mathscr{A}, W)$, which is injective on $H^\bullet(\mathrm{Hom}(\mathbb{L}, U))$ for any U.*

An important feature is that the Landau-Ginzburg superpotential W constructed in this way is automatically a central element in $\mathscr{A}$. In particular we can make sense of $\mathscr{A}/\langle W \rangle$ as a hypersurface singularity defined by 'the zero set' of W.

THEOREM 1.2 (Theorem 3.10 and 6.6). *$W \in \mathscr{A}$ is a central element.*

$$m_k : \mathscr{C}[1](A_0, A_1) \otimes \cdots \mathscr{C}[1](A_{k-1}, A_k) \to \mathscr{C}[1](A_0, A_k)$$

for $A_i \in Ob(\mathscr{C})$, $k = 0, 1, 2, \cdots$. Each A_∞-operation m_k is assumed to respect the filtration, and satisfies the A_∞-equation: for $v_i \in \mathscr{C}[1](A_i, A_{i+1})$,

$$\sum_{k_1 + k_2 = n+1} \sum_i (-1)^{\epsilon_1} m_{k_1}(v_1, \cdots, m_{k_2}(v_i, \cdots, v_{i+k_2-1}), v_{i+k_2}, \cdots v_n) = 0.$$

We denote by $|v|$ the degree of v and by $|v|' = |v| - 1$ the shifted degree of v_j. Then $\epsilon_1 = \sum_{j=1}^{i-1} (|v_j|')$.

Let A be an A_∞-algebra. When $m_0 \neq 0$, m_1 may not define a differential, which can be seen in the following A_∞-equation:

$$0 = m_1^2(v_1) + m_2(m_0, v_1) + (-1)^{|v|'} m_2(v_1, m_0) \tag{2.1}$$

The obstruction and deformation theory of such A_∞-algebras have been studied by Fukaya-Oh-Ohta-Ono[**FOOO09**], who introduced the notion of weak bounding cochains (weak Maurer-Cartan elements).

For this purpose, recall that an element $\mathbf{1}_A \in \mathscr{C}^0(A, A)$ is called a *unit* if it satisfies

$$\begin{cases} m_2(\mathbf{1}_A, v) = v & v \in \mathscr{C}(A, A') \\ (-1)^{|w|} m_2(w, \mathbf{1}_A) = w & w \in \mathscr{C}(A', A) \\ m_k(\cdots, \mathbf{1}_A, \cdots) = 0 & \text{otherwise.} \end{cases}$$

Note that if m_0 is a constant multiple of a unit, then the latter two terms of (2.1) vanishes by the property of a unit. This happens for A_∞-algebras of monotone Lagrangians. In general, a boundary deformation of a given A_∞-algebra via an weak Maurer-Cartan element b can be used to define a deformed A_∞-algebra $\{m_k^b\}$ such that m_0^b becomes a multiple of a unit. Let us use the notation

$$b^l = \underbrace{b \otimes \cdots \otimes b}_{l}, \;\; e^b := 1 + b + b^2 + b^3 + \cdots.$$

DEFINITION 2.2. An element $b \in F^+\mathscr{C}^1(A, A)$ is a *weak Maurer-Cartan* element if $m(e^b) := \sum_{k=0}^\infty m_k(b, \cdots, b)$ is a multiple of the unit, i.e.

$$m(e^b) = PO(A, b) \cdot \mathbf{1}_A, \;\; \text{for some } PO(A, b) \in \Lambda$$

Denote by $\widetilde{\mathscr{M}}_{weak}^+(A)$ the set of all weak Maurer-Cartan elements.

DEFINITION 2.3. Given $b \in F^+\mathscr{C}^1(A, A)$, we define the deformed A_∞-operation m_k^b as

$$m_k^b(v_1, \cdots, v_k) = \sum_{l_0, \cdots, l_k \geq 0} m_{k + l_0 + \cdots + l_k}(b^{l_0}, v_1, b^{l_1}, v_2, \cdots, v_k, b^{l_k})$$

$$= m(e^b, v_1, e^b, v_2, \cdots, e^b, v_k, e^b).$$

Then $\{m_k^b\}$ defines an A_∞-algebra. In general, given $b_0, \cdots, b_k \in F^+\mathscr{C}^1(A, A)$, we define

$$m_k^{b_0, \cdots, b_k}(v_1, \cdots, v_k) = m(e^{b_0}, v_1, e^{b_1}, v_2, \cdots, e^{b_{k-1}}, v_k, e^{b_k}).$$

Note that we have $m_k^b = m_k^{b, b, \cdots, b}$.

Given a weak Maurer-Cartan element b, we have $m_0^b = PO(A, b) \cdot \mathbf{1}_A$, and one can check that $(m_1^b)^2 = 0$. And if $PO(A, b_0) = PO(A, b_1)$, we have $(m_1^{b_0, b_1})^2 = 0$.

In this section we perform a base change of an A_∞-algebra A. Originally A is over the Novikov ring, and we enlarge the base to be a noncommutative algebra. This is an important step for deformations.

Let K be a noncommutative algebra over Λ_0. Consider a filtered A_∞-algebra $(A, \{m_k\})$ over Λ_0. We will consider an induced A_∞-algebra structure on the completed tensor product $K \widehat{\otimes}_{\Lambda_0} A$ where we take a completion with respect to the energy, namely the power of the formal variable $\mathbf{T}$.

DEFINITION 2.6. We define an A_∞-structure on

$$\tilde{A}_0 := K \widehat{\otimes}_{\Lambda_0} A \tag{2.2}$$

For $f_i \in K, e_i \in A$ $i = 1, \cdots, k$, the A_∞-operation is defined as

$$m_k(f_1 e_1, f_2 e_2, \cdots, f_k e_k) := f_k f_{k-1} \cdots f_2 f_1 \cdot m_k(e_1, \cdots, e_k). \tag{2.3}$$

Then we extend it linearly to define the A_∞-structure on $\tilde{A}_0$ and also tensor over Λ to get the A_∞-structure on $\tilde{A} := \tilde{A}_0 \otimes \Lambda$.

LEMMA 2.9. *$(\tilde{A}, \{m_k\})$ satisfies the A_∞-equation.*

PROOF. From linearity, it is enough to prove it when inputs are multiples of basis elements. Namely, we consider the expansion of $m(\widehat{m}(f_1 e_1, f_2 e_2, \cdots, f_n e_n))$ which are given by

$$\sum_{k_1 + k_2 = n+1} m_{k_1}(f_1 e_1, \cdots, m_{k_2}(f_{j+1} e_{j+1}, \cdots), \cdots, f_n e_n).$$

Here, $\widehat{m}$ is the coderivation corresponding to m. From the A_∞-equation of A, we have

$$f_n f_{n-1} \cdots f_2 f_1 \sum_{k_1 + k+2 = n+1} m_{k_1}(e_1, \cdots, m_{k_2}(e_{j+1}, \cdots), \cdots, e_n) = 0.$$

□

The unit $\mathbf{1}_A$ of A is also the unit of $\tilde{A}$. Thus the noncommutative version of the weak Maurer-Cartan equation makes sense.

6.2. Mirror construction

As in Section 2.3, first we perform a base change for the A_∞-algebra:

$$\tilde{A}^{\mathbb{L}} := \Lambda Q^{\mathbb{L}} \widehat{\otimes}_{\Lambda^{\oplus}} \mathrm{CF}(\mathbb{L},\mathbb{L}).$$

Due to the bimodule structure, an expression $pX_e := p \otimes X_e$ for a path p and $X_e \in \mathrm{CF}(L_i, L_j)$ is non-zero if and only if $t(p) = i$. We use Definition 2.6 to extend the A_∞-structure on $\mathrm{CF}(\mathbb{L},\mathbb{L})$ to $\tilde{A}^{\mathbb{L}}$.

Denote the formal variable in ΛQ associated to each arrow e of Q by x_e, and denote the corresponding odd-degree immersed generator in $\mathrm{CF}(\mathbb{L},\mathbb{L})$ by X_e. Now take the linear combination

$$b = \sum_e x_e X_e \in \tilde{A}^{\mathbb{L}}.$$

As in Definition 3.1, $\deg x_e := 1 - \deg X_e$ so that b has degree one. In particular $\deg x_e$ is even. We define nc-weak Maurer-Cartan relations in the following way (assuming the Fukaya category $\mathscr{C}$ is unital).

DEFINITION 6.5. The coefficients P_f of the even-degree generators X_f of $\mathrm{CF}(\mathbb{L},\mathbb{L})$ (other than the fundamental classes $\mathbf{1}_{L_i}$) in

$$m_0^b = m(e^b) = \sum_i W_i \mathbf{1}_{L_i} + \sum_f P_f X_f$$

are called the nc-weak Maurer-Cartan relations. Let R be the completed two-sided ideal generated by P_f. Then define the noncommutative ring

$$\mathscr{A} := \Lambda Q / R.$$

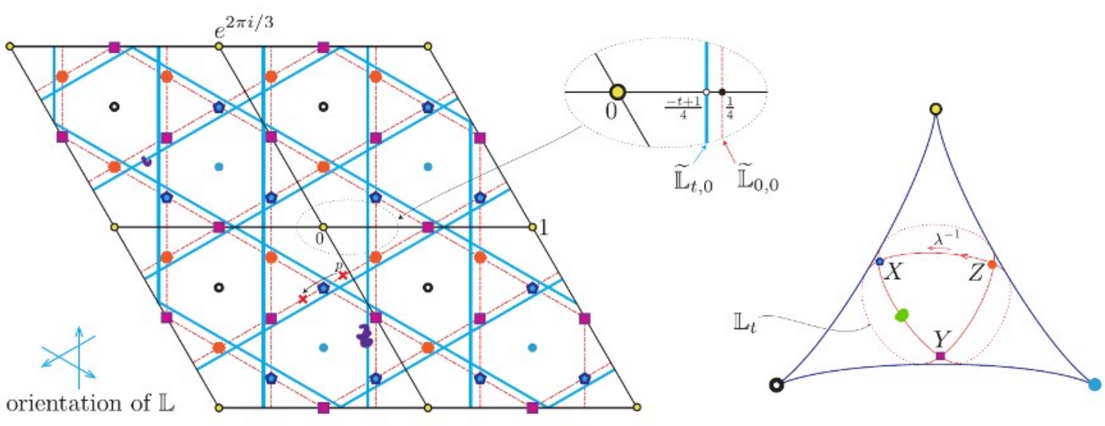
$W_{\mathbb{L}} = \sum_i W_i$ is called the worldsheet superpotential of $\mathbb{L}$.

We regard $\mathscr{A}$ to be the space of noncommutative weakly unobstructed deformations of $\mathbb{L}$. Instead of working on $\tilde{A}^{\mathbb{L}} = \Lambda Q^{\mathbb{L}} \widehat{\otimes}_{\Lambda^{\oplus}} \mathrm{CF}(\mathbb{L},\mathbb{L})$, we define $\mathscr{A} = \Lambda Q^{\mathbb{L}} / R$ as above and work on

$$A^{\mathbb{L}} = \mathscr{A} \widehat{\otimes}_{\Lambda^{\oplus}} \mathrm{CF}(\mathbb{L},\mathbb{L}).$$

Now, the induced A_∞-structure on $A^{\mathbb{L}}$ satisfies

$$m_0^b = W_{\mathbb{L}} \cdot \mathbf{1}_{\mathbb{L}} = \sum_i W_i \cdot \mathbf{1}_{L_i} \in \mathscr{A} \widehat{\otimes}_{\Lambda^{\oplus}} \mathrm{CF}(\mathbb{L},\mathbb{L}) \tag{6.2}$$



(a) (b)

THEOREM 5.3 ([**CHL17**]). *When $\mathbb{L}_0$ is equipped with a non-trivial spin structure, $b = xX + yY + zZ$ is a weak Maurer-Cartan solution for any $x, y, z \in \mathbb{C}$. The mirror LG superpotential W_0, after a rescaling on x, y, z, takes the form*

$$W_0 = x^3 + y^3 + z^3 - \sigma(q_{\text{orb}})xyz \tag{5.2}$$

where $q_{\text{orb}} = \mathbf{T}^{\omega(\mathbb{P}^1_{3,3,3})}$ is the Kähler parameter of $\mathbb{P}^1_{3,3,3}$ and $\sigma(q_{\text{orb}})$ is the inverse mirror map

$$\sigma(q_{\text{orb}}) = -3 - \left(\frac{\eta(q_{\text{orb}})}{\eta(q_{\text{orb}}^9)}\right)^3. \tag{5.3}$$

η above denotes the Dedekind eta function.

Weakly unobstructedness of $(\mathbb{L}_0, \lambda = -1)$ is mainly due to the symmetry of $\mathbb{L}_0$ under the anti-symplectic involution together with certain sign computations. The potential W_0 was computed by counting infinite series of triangles passing through a given point class. (In fact, with our new formulation of non-standard spin structure, the coordinate change $y \to \tilde{y}$ in [**CHL17**] is not necessary. More details will be given below.)

THEOREM 5.1. *There is a T^2-family $(\mathbb{L}_t, \lambda)$ of Lagrangians decorated by flat $U(1)$ connections in $\mathbb{P}^1_{3,3,3}$ for $(t-1) \in \mathbb{R}/2\mathbb{Z}$ and $\lambda \in U(1)$ whose corresponding generalized mirror $(\mathscr{A}_{(\lambda,t)}, W_{(\lambda,t)})$ satisfies the following.*

(1) The noncommutative algebras $\mathscr{A}_{(\lambda,t)}$ are Sklyanin algebras, which are of the form

$$\mathscr{A}_{(\lambda,t)} := \frac{\Lambda < x, y, z >}{(axy + byx + cz^2, ayz + bzy + cx^2, azx + bxz + cy^2)} \tag{5.1}$$

for $a = a(\lambda, t), b = b(\lambda, t), c = c(\lambda, t) \in \Lambda$. We have $\mathscr{A}_0 := \mathscr{A}_{(-1,0)} = \Lambda[x, y, z]$.

(2) $W_{(\lambda,t)}$ lies in the center of $\mathscr{A}_{(\lambda,t)}$ for all (λ, t). We denote $W_0 = W_{(-1,0)}$.

(3) The coefficients $(a : b : c)$ are given by theta functions, which define an embedding $T^2 \to \mathbb{P}^2$ onto the mirror elliptic curve

$$\check{E} = \{(a : b : c) \in \mathbb{P}^2 \mid W_0(x, y, z) = 0\}.$$

(4) For each (λ, t), there exists a $\mathbb{Z}_2$-graded A_∞-functor

$$\mathscr{F}^{(\mathbb{L}_t,\lambda)} : \mathrm{Fuk}(\mathbb{P}^1_{3,3,3}) \to \mathrm{MF}(\mathscr{A}_{(\lambda,t)}, W_{(\lambda,t)}).$$

Upstairs there is a $\mathbb{Z}$-graded A_∞-functor

$$\mathscr{F}^{(\tilde{\mathbb{L}}_t,\lambda)} : \mathrm{Fuk}^{\mathbb{Z}}(E) \to \mathrm{MF}^{\mathbb{Z}}(\hat{\mathscr{A}}_{(\lambda,t)}, \hat{W}_{(\lambda,t)}).$$

When $t = 0, \lambda = -1$, they give derived equivalences.

(5) The graded noncommutative algebra $\mathscr{A}_{(\lambda,t)}/\langle W_{(\lambda,t)}\rangle$ is a twisted homogeneous coordinate ring of $\check{E}$.

(6) The family of noncommutative algebras $\mathscr{A}_{(\lambda,t)}/\langle W_{(\lambda,t)}\rangle$ near $t = 0, \lambda = -1$ gives a quantization of the affine del Pezzo surface defined by $W_0(x, y, z) = 0$ in $\mathbb{C}^3$.

5.5. Relation to the quantization of an affine del Pezzo surface

Recall that deformation quantization of a (commutative) Poisson algebra is a formal deformation into a noncommutative associative unital algebra whose commutator in the first order is the Poisson bracket.

For any $\phi \in \mathbb{C}[x, y, z]$, the following brackets on coordinated functions extends to a Poisson struncture on $\mathbb{C}[x, y, z]$:

$$\{x, y\} = \frac{\partial \phi}{\partial z}, \quad \{y, z\} = \frac{\partial \phi}{\partial x}, \quad \{z, x\} = \frac{\partial \phi}{\partial y}. \tag{5.15}$$

One can check that ϕ itself Poisson commute with any other element, and hence the above Poisson structure descend to the quotient $\mathbb{C}[x, y, z]/(\phi)$ by the principal ideal generated by ϕ, which is denoted as $\mathscr{B}_\phi$.

Our theory provides the quantization of the affine del Pezzo surface in the sense of [**EG10**]. We write $v = u - u_0$ so that now $v = 0$ corresponds to the commutative point (see the last paragraph of 5.2).

THEOREM 5.14. *The family of noncommutative algebra $\mathscr{A}_v/(W_v)$ near $v = 0$ gives a quantization of the affine del Pezzo surface, given by the mirror elliptic curve equation $W_0(x, y, z) = 0$ in $\mathbb{C}^3$ in place of ϕ in* (5.15).

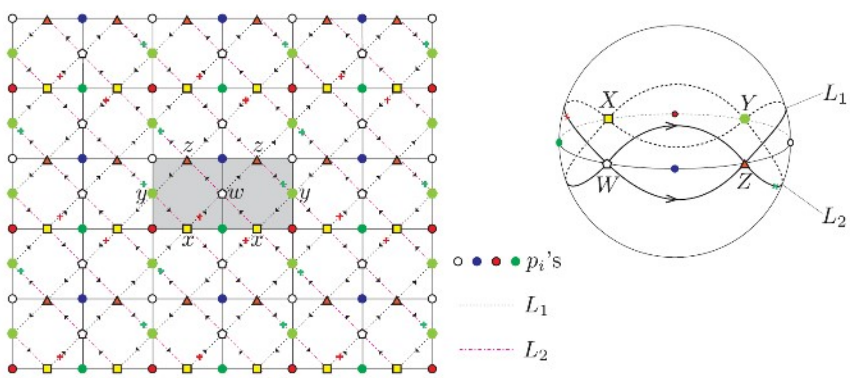
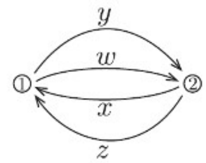


FIGURE 12. The universal cover of $\mathbb{P}^1_{2,2,2,2}$. The bold lines show the lifts of the equator of $\mathbb{P}^1_{2,2,2,2}$, and the dotted lines show the lifts of the Lagrangians L_1 and L_2.

THEOREM 8.1. *The generalized mirror of $E/\mathbb{Z}_2 = \mathbb{P}^1_{2,2,2,2}$ corresponding to $\mathbb{L}_0$ is given by $(\mathscr{A}_0, W_0)$ where*



(1) *Q is the directed graph with two vertices v_1, v_2, two arrows $\{y, w\}$ from v_1 to v_2 and two arrows $\{x, z\}$ from v_2 to v_1.*

(2) *$\mathscr{A}_0 = \mathscr{A}(Q, \Phi_0)$ is the noncommutative resolution of the conifold $\frac{\Lambda Q}{(\partial \Phi_0)}$, where $\Phi_0 := xyzw - wzyx$.*

(3)

$$W_0 = \phi(q_{\text{orb}}^{\frac{1}{4}})((xy)^2+(xw)^2+(zy)^2+(zw)^2+(yx)^2+(wx)^2+(yz)^2+(wz)^2)+\psi(q_{\text{orb}}^{\frac{1}{4}})(xyzw+wzyx)$$

lies in the center of $\mathscr{A}_0$, where ϕ and ψ are given in Theorem 8.7 and $q_{\text{orb}} = \exp\left(-\int_{\mathbb{P}^1_{2,2,2,2}} \omega\right)$ is the Kähler parameter.

(4) *$\psi(q_{\text{orb}}^{\frac{1}{4}})\left(\phi(q_{\text{orb}}^{\frac{1}{4}})\right)^{-1}$ equals to the mirror map of $\mathbb{P}^1_{2,2,2,2}$.*

(5) *We have $\mathbb{Z}_2$-graded, and $\mathbb{Z}$-graded A_∞-functors*

$$\mathscr{F}^{\mathbb{L}_0} : \text{Fuk}(\mathbb{P}^1_{2,2,2,2}) \to \text{MF}(\mathscr{A}_0, W_0), \quad \mathscr{F}^{\mathbb{L}_0} : \text{Fuk}^{\mathbb{Z}}(E) \to \text{MF}^{\mathbb{Z}}(\mathscr{A}_0, W_0).$$

THEOREM 8.2. *There is a T^2-family $(\mathbb{L}_t, \lambda)$ of Lagrangians decorated by flat $U(1)$-connections for $(t-1) \in \mathbb{R}/2\mathbb{Z}$ and $\lambda \in U(1)$ such that*

(1) *the corresponding mirror noncommutative algebras $\mathscr{A}_{(\lambda,t)}$ takes the form*

$$\mathscr{A}_{(\lambda,t)} := \frac{\Lambda Q}{(\partial \Phi_{(\lambda,t)})}, \Phi = a(\lambda,t)\,xyzw + b(\lambda,t)\,wzyx + \frac{1}{2}c(\lambda,t)\,((wx)^2+(yz)^2) + \frac{1}{2}d(\lambda,t)\,((xy)^2+(zw)^2)$$

($\mathscr{A}_{(1,0)}$ is the same as $\mathscr{A}_0$ in Theorem 8.1 (2))

(2) *Coefficients $(a:b:c:d)$ defines an embedding $T^2 \to \mathbb{P}^3$ onto the complete intersection of two quadrics given as follows. For $[x_1, x_2, x_3, x_4] \in \mathbb{P}^3$ and $\sigma = \frac{\psi}{\phi}$,*

$$x_1 x_3 = x_2 x_4$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + \sigma x_1 x_3 = 0.$$

This is isomorphic to the mirror elliptic curve $\check{E}$ given by Hesse cubic in Theorem 5.1.

(3) *The family of noncommutative algebra $\mathscr{A}_{\lambda,t}/(W_{\lambda,t})$ near $t = 0, \lambda = 1$ gives a quantization of the complete intersection given by above two quadratic equations in $\mathbb{C}^4$ in the sense of [**EG10**].*

The subalgebra $\underline{\mathscr{A}}$ of $\mathscr{A}$ is obviously the quotient of $\mathbb{C}\{x_4, x_3, x_2, x_1\}$ by the ideal generated by these eight relations. Recall that a, b, c, d are functions in

$$u = -s - \tau\frac{t}{2} - \frac{1}{4}$$

(and $\lambda = e^{2\pi \mathbf{i} s}$). To emphasize the dependence of $\underline{\mathscr{A}}$ in u, we write $\underline{\mathscr{A}}_u$ from now on.

Note that $\underline{\mathscr{A}}_{u_0}$ represents the commutative conifold since $a(u_0) = -b(u_0)$ and $c(u_0) = d(u_0) = 0$:

$$\underline{\mathscr{A}}_{u_0} \cong \mathbb{C}[x_1, x_2, x_3, x_4]/x_1x_3 - x_2x_4. \tag{8.13}$$

Thus one may view $\underline{\mathscr{A}}_u$ as a noncommutative conifold. Let

$$f = x_1x_3 - x_2x_4$$

be the defining equation of the conifold.

If we are given another function g on $\mathbb{C}^4$, we can define a Poisson structure in the following way.

$$\{x_i, x_j\} = \zeta \frac{dx_i \wedge dx_j \wedge df \wedge dg}{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4} = \zeta\left(\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial x_l} - \frac{\partial g}{\partial x_k}\frac{\partial f}{\partial x_l}\right). \tag{8.14}$$

where (i, j, k, l) is equivalent to $(1,2,3,4)$ up to an even permutation. ζ in (8.14) will be some constant in our case though it could be a more complicated function in general. Such a structure is in fact a special case of certain higher brackets among functions called the *Nambu bracket*. (See for e.g. [**OR02**].) Choice of g will be fixed shortly.

One can check that (8.14) satisfies a Jacobi relation, and f and g lie in the Poisson center. Therefore it descends to the quotient algebra $\mathscr{B}_{f,g} = \mathbb{C}[x_1, x_2, x_3, x_4]/\langle f, g\rangle$ which is a coordinate ring of a hypersurface in the conifold defined by $g = 0$. Now we provide some relation between our $\underline{\mathscr{A}}_u/W_u$ and the deformation quantization of $\mathscr{B}_{f,g}$ associated with the Poisson structure (8.14).

First, we make the following specific choice of the second function g in (8.14). Restricted to the loops based at the first vertex of Q, world sheet potential $(W_{(\lambda,t)})_1$ (8.10)

THEOREM 8.10. *The family of noncommutative algebra $\underline{\mathscr{A}}_v/(W_v)$ near $v = 0$ gives a quantization of the complete intersection given by two quadratic equations $f = 0$ and $g = 0$ in $\mathbb{C}^4$ in the sense of* [**EG10**].

We remark that $\{f = 0\} \cap \{g = 0\}$ defines the mirror elliptic curve in $\mathbb{P}^3$ after projectivization. As in Theorem 5.15 [**ATdB90, Ste97**], we expect that $\underline{\mathscr{A}}_v/(W_v)$ is given as a twisted homogeneous coordinate ring of the mirror elliptic curve embedded in the quadric $\{x_1x_3 = x_2x_4\} \subset \mathbb{P}^3$ (which can be identified as $\mathbb{P}^1 \times \mathbb{P}^1$) by the line bundle $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1}(1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(1)$.

PROOF. As $v \to 0$, we have

$$\underline{\mathscr{A}}_v/(W_v) \to \mathbb{C}[x_1, x_2, x_3, x_4]/\langle f, g\rangle.$$

So it remains to check that the first order terms of the commutators for A_v are equivalent to the Poisson bracket induced by f and g with the formula (8.14). The first equation in (8.12) gives rise to

$$r(x_3x_4 - x_4x_3) = v\left(a'(0)x_4x_3 + b'(0)x_3x_4 + c'(0)x_3x_2 + d'(0)x_4x_1\right) + o(v),$$

where $r := a(0) = -b(0) \in \mathbb{C}$. Thus we obtain

$$\lim_{v\to 0} \frac{x_3x_4 - x_4x_3}{v} = \frac{1}{r}\left((a'(0) + b'(0))x_3x_4 + c'(0)x_3x_2 + d'(0)x_4x_1\right) \tag{8.16}$$

since x_3 and x_4 commute at $v = 0$ (and so do other variables).

On the other hand, the Poisson bracket between x_3 and x_4 from (8.14) is given by

$$\{x_3, x_4\} = \zeta\left(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2}\right) = \zeta\left(\psi x_3x_4 + 2\phi(x_3x_2 + x_4x_1)\right),$$

which is proportional to the right hand side of (8.16) since $a'(0) + b'(0) = \zeta\psi$ and $c'(0) = d'(0) = 2\zeta\phi$. Similarly, the first order commutator between x_i and x_{i+1} for other i's agrees with the Poisson bracket of the corresponding variables.

DEFINITION 9.1 (Extended quiver). Let $\{X_f\}$ be the set of even-degree immersed generators. Moreover take a Morse function on each L_i and let $\{T_g\}$ be the set of their critical points *other than* the maximum points (which correspond to the fundamental classes $\mathbf{1}_{L_i}$). The extended quiver $\tilde{Q}$ is defined to be a directed graph whose vertices are one-to-one corresponding to $\{L_i\}$, and whose arrows are one-to-one corresponding to $\{X_e\} \cup \{X_f\} \cup \{T_g\}$.

DEFINITION 9.3. Take

$$\tilde{b} = \sum_e x_e X_e + \sum_f x_f X_f + \sum_g t_g T_g.$$

Define $\deg x_e = 1 - \deg X_e$, $\deg x_f = 1 - \deg X_f$ and $\deg t_g = 1 - \deg T_g$ such that $\tilde{b}$ has degree one.

The deformation parameter $\deg x_e$ is always even, but the other parameters may not be. Hence we need to modify the sign rule of (2.3) in the definition 2.6 to the following.

$$m_k(f_1e_{i_1}, f_2e_{i_2}, \cdots, f_ke_{i_k}) := (-1)^\epsilon f_kf_{k-1}\cdots f_2f_1 \cdot m_k(e_{i_1}, \cdots, e_{i_k}). \tag{9.1}$$

where ϵ is obtained from the usual Koszul sign convention. For example, if f_j is the dual variable of e_{i_j} for all $j = 1, \dots k$, then we have

$$m_k(f_1e_{i_1}, f_2e_{i_2}, \cdots, f_ke_{i_k}) := (-1)^{\sum_{j=1}^k \deg f_j} f_kf_{k-1}\cdots f_2f_1 \cdot m_k(e_{i_1}, \cdots, e_{i_k}), \tag{9.2}$$

since one can pull out $f_k, f_{k-1}, \dots, f_1$ one at a time and by assumption $f_je_{i_j}$ will have (shifted) degree 0. One can check that this defines an A_∞-structure.

Then $m_0^{\tilde{b}}$ will additionally have odd-degree part, i.e., it is of the form

$$m_0^{\tilde{b}} = \sum_i W_i \mathbf{1}_{L_i} + \sum_e P_e X_e + \sum_f P_f X_f + \sum_g P_g T_g.$$

graded ✓

$$m_0^{\tilde{b}} = \sum_i W_i \mathbf{1}_{L_i} + \sum_e P_e X_e + \sum_f P_f X_f + \sum_g P_g T_g.$$

graded

The constructions in previous chapters generalize to this setting, to give the non-commutative ring A modulo the relations (P_e, P_f, P_g), and an extended A_∞ functor to matrix factorizations of an extended potential function $\widetilde{W}$.

Alternatively, we explain a construction which gives a curved dg-algebra $(A, d, \widetilde{W})$, and an A_∞-functor to the category of curved dg-modules over $(A, d, \widetilde{W})$.

Better for homotopy theory: working with complexes rather than cohomology spaces.

We will construct a curved dg-algebra from the extended deformations.

DEFINITION 9.4 ([**KL03**]). A curved dg-algebra is a triple (A, d, F), where A is a graded associative algebra, d is a degree-one derivation of A, and $F \in A$ is a degree 2 element such that $dF = 0$ and $d^2 = [F, \cdot] = F(\cdot) - (\cdot)F$.

If $d^2 = 0$ (or equivalently F is central), a curved dg-algebra structure gives rise to a noncommutative Landau-Ginzburg model on its d-cohomology with the cohomology class $[F]$ as a (worldsheet) superpotential. If the dg-algebra is formal, then dg-modules over A (Definition 9.11 below) boils down to matrix factorizations of $[F]$ over $H(A)$. In this section we don't take this assumption and work with dg-modules.

Now take the path algebra $\Lambda\tilde{Q}$. It carries a natural curved dg structure as follows.

DEFINITION 9.5. Define the extended worldsheet superpotential to be $\tilde{W} = \sum_i \tilde{W}_i$, where $\tilde{W}_i$ are the coefficients of $\mathbf{1}_{L_i}$ in $m_0^{\tilde{b}}$. Define a derivation d of degree one on $\Lambda\tilde{Q}$ by

$$d(a_e x_e + b_f x_f + c_g t_g) = a_e P_e + b_f P_f + c_g P_g$$

$dx_e = P_e$
$dx_f = P_f$
$dt_g = P_g$

for $a_e, b_f, c_g \in \Lambda$, and extend it using Leibniz rule. (Our sign convention is $d(xy) = x\,dy + (-1)^{\deg y}(dx)y$.)

Then the A_∞-relations for $\mathbb{L}$ can be rewritten as follows.

PROPOSITION 9.6. *$d\tilde{W} = 0$ and $d^2 = [\tilde{W}, \cdot]$. In other words $(\Lambda\tilde{Q}, d, \tilde{W})$ is a curved dg algebra.*

REMARK 9.7. It is well-known (see [**KS09**]) that A_∞-algebra structure on V is equivalent to a codifferential $\hat{d}$ on the tensor coalgebra TV. And if V is finite dimensional (as in our case), then one can take its dual algebra $(TV)^*$ with a differential $\tilde{d}$ (with $\tilde{d}^2 = 0$). We may regard $\{x_e, x_f, x_g\}$ as dual generators of the dual algebra $(TV)^*$. There is one more generator $x_{\mathbf{1}}$ that we have suppressed in the previous formulation. Turning it on simply results in adding a multiple of unit in $\tilde{W}$. The desired identities can be obtained from $\tilde{d}^2 = 0$ by setting $x_{\mathbf{1}} = 0$. But we show the direct proof below due to signs. These two formulations are equivalent.

In general it is difficult to write down $(\Lambda\tilde{Q}, d, \tilde{W})$ explicitly since it requires computing all m_k's for $\mathbb{L}$. On the other hand, for Calabi-Yau threefolds with grading assumptions (see below), a lot of terms automatically vanish by dimension reason, and $(\Lambda\tilde{Q}, d, \tilde{W})$ can be recovered from the quiver Q with potential Φ (and $\tilde{W} = 0$). The resulting dg algebra $(\Lambda\tilde{Q}, d)$ is known as Ginzburg algebra.

DEFINITION 9.9 ([**Gin**]). Let (Q, Φ) be a quiver with potential. The Ginzburg algebra associated to (Q, Φ) is defined as follows. Let $\tilde{Q}$ be the doubling of Q by adding a dual arrow $\bar{e}$ (in reversed direction) for each arrow e of Q and a loop based at each vertex of Q. Define the grading on the path algebra $\Lambda\tilde{Q}$ as

$$\deg x_e = 0, \deg x_{\bar{e}} = -1, \deg t_i = -2$$

where t_i corresponds to the loop at a vertex. Define the differential d by

$$dt_i = \sum_e \pi_{v_i} \cdot [x_e, x_{\bar{e}}] \cdot \pi_{v_i}, \quad dx_{\bar{e}} = \partial_{x_e} \Phi, \quad dx_e = 0$$

where π_{v_i} denotes the constant path at the vertex v_i. Then the dg algebra $(\Lambda\tilde{Q}, d)$ is called the *Ginzburg algebra.*

PROPOSITION 9.10. Suppose X is a Calabi-Yau threefold, L_i are graded Lagrangian called the *Ginzburg algebra*.

PROPOSITION 9.10. Suppose X is a Calabi-Yau threefold, L_i are graded Lagrangian spheres (equipped with a Morse function with exactly one maximum point and one minimum point) such that deg X equals to either one or two for all immersed generators X. Then $(\Lambda\tilde{Q}, d, \tilde{W} = 0)$ produced from the above construction is the Ginzburg algebra associated to (Q, Φ).

Assume cyclicity: $m_0^b = \sum (\partial_{x_e} \Phi) X_{\bar{e}}$. $\Phi = \overbrace{\left(m_k(\sum x_e X_e, \ldots, \sum x_e X_e), \sum x_e X_e\right)}^{m_0^b}$

DEFINITION 9.11. A curved dg-module over a curved dg-algebra (A, d, F) is a pair (M, d_M) where M is a graded A-module and d_M is a degree one linear endomorphism of M such that

$$d_M(am) = a(d_M m) + (-1)^{\deg m}(da)m \tag{9.4}$$

and $d_M^2 = F$.

Conclusion:

We have learnt the following topics about quivers:

1. Quiver representations (modules of path algebra)
2. Classification of indecomposable representations (Gabriel and Katz)
3. Moduli space of stable quiver representations [Kings]
4. Quiver moduli associated to complex varieties (to understand the derived category, or to provide ambient space of embeddings)
5. Cohomology: resolutions of quiver modules to compute Ext and HH
6. Framed moduli as iterated Grassmannian bundles [Reineke]
7. Nakajima quiver variety
8. Quiver algebra constructed as mirror of symplectic geometry

There are still several topics of quiver variety that we have not got into, such as

- relation with cluster variety
- applications like persistence theory

Some of your reports are introducing these topics.

Wish you have enjoyed the course!